Optimal bounds and blow-up criteria for a semi-linear accretive wave equation

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Abstract. In this paper we consider the semi-linear wave equation: $u_{tt} - \Delta u = u_t|u_t|^{p-1}$ in $\mathbb{R}^N$ where $1 < p \le 1 + \frac{2}{N-1}$ and $p < 3$ if $N = 1$, $p \ne 3$ if $N = 2$. We give an energetic criteria and optimal lower bound for blowing up solutions of this equation.

1 Introduction

We consider the following semi-linear wave equation:

$$\begin{cases} u_{tt} - \Delta u = u_t\,|u_t|^{p-1} & t \in [0,T),\ x \in \mathbb{R}^N \\ u\,(x,0) = u_0 \in H^1_{loc,\,u}\left(\mathbb{R}^N\right) & \\ u_t\,(x,0) = u_1 \in L^2_{loc,\,u}\left(\mathbb{R}^N\right), & \end{cases} \tag{1}$$

where $1 < p \le 1 + \frac{2}{N-1}$ and $p < 3$ if $N = 1$, $p \ne 3$ if $N = 2$, and where

$$L^2_{loc,\,u}\left(\mathbb{R}^N\right) = \left\{ u\colon \mathbb{R}^N \to \mathbb{R};\ \|u\|_{L^2_{loc,\,u}} := \sup_{x_0 \in \mathbb{R}^N} \int_{|x-x_0|\le 1} |u(x)|^2 dx\ < \infty \right\}$$

and

$$H^1_{loc,\,u}\left(\mathbb{R}^N\right) := \left\{ u \in L^2_{loc,\,u}\left(\mathbb{R}^N\right);\ |\nabla u| \in\ L^2_{loc,\,u}\left(\mathbb{R}^N\right) \right\}.$$

A very rich literature has been done on the non linear equation

$$u_{tt} - \Delta u = a u_t\,|u_t|^{p-1} + b u|u|^{q-1} \tag{2}$$

with a and b are real numbers. When $a \le 0$ and $b = 0$ then the damping term $au_t|u_t|^{p-1}$ assume global existence for arbitrary data (see, for instance, Harraux and Zuazua [7] and Kopackova [9]). When $a \le 0$, $b > 0$ and $p > q$ then one can cite, for instance, Levine [10] and Georgiev and Todorova [4], that show the existence of global solutions (in time) under negative energy condition. When $a \le 0$, $b > 0$ and $q > p$, or when $a \le 0$, $b > 0$ and $p = 1$ then one can cite [4] and Messaoudi [12] where they show finite time blowing up solutions under sufficiently large negative energy of the initial condition.

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The first to consider the case $a > 0$ was Haraux [6] (with $b = 0$ on bounded domain), who construct blowing up solutions for arbitrary small initial data. See also Jazar and Kiwan [8] and the references therein for the same equation (1) on bounded domain. We refer to Levine, Park and Serrin [11] and the references therein for the whole space-case $\mathbb{R}^N$. Finally, we refer to Haraux [6], Souplet [14, 15] and Jazar and Souplet [3] concerning the ODE case.

In this paper, we consider solutions u of (1) that blows-up in finite time $T > 0$ in the space $H^1_{loc,\,u}\left(\mathbb{R}^N\right) \times L^2_{loc,\,u}\left(\mathbb{R}^N\right)$. Our aim is to study the blow-up behavior of $u(t)$ as $t \uparrow T$.
Following the work of Antonini and Merle [2], we compare the growth of u_t and k, the solution of the simplest associated ODE: $k_{tt} = k_t|k_t|^{p-1}$. Nevertheless, the presence of the force term $u_t|u_t|^{p-1}$ makes the work more complicated. To remedy this difficulty, and inspired by the work of Rivera and Fatori [13], we rewrite (1) as

$$\begin{cases} u_{tt} - \int_0^t \Delta u_t\left(\tau\right) d\tau - \Delta u_0 = u_t\left|u_t\right|^{p-1}, \quad t \in [0,T),\, x \in \mathbb{R}^N, \\ u\left(x,0\right) = u_0\left(x\right) \in H^1_{loc,\,u}\left(\mathbb{R}^N\right), \\ u_t\left(x,0\right) = u_1\left(x\right) \in L^2_{loc,\,u}\left(\mathbb{R}^N\right). \end{cases} \tag{3}$$

Then, putting

$$v\left(x,t\right) = u_t\left(x,t\right) \tag{4}$$

in (3), we obtain the following integro differential PDE

$$\begin{cases} v_t - \int_0^t \Delta v\left(\tau\right) d\tau - \Delta u_0\left(x\right) = v\left|v\right|^{p-1} \quad t \in [0,T),\, x \in \mathbb{R}^N, \\ v\left(x,0\right) = u_1\left(x\right) =: v_0 \in L^2_{loc,\,u}\left(\mathbb{R}^N\right). \end{cases} \tag{5}$$

Now, we introduce $w := u_t/k_t$, where $k_t(t) := \kappa(T-t)^{-\beta}$ with $\beta := \frac{1}{p-1}$ and $\kappa := \beta^\beta$ (see [2, 1]), and we use the classical self-similar transformation (see [5]): for $a \in \mathbb{R}^N$ and $T' > 0$:

$$y = \frac{x-a}{\sqrt{T'-t}}, \quad s = -\log\left(T'-t\right), \quad v\left(t,x\right) = \frac{1}{\left(T'-t\right)^\beta} w_{T',\,a}\left(s,y\right) \tag{6}$$

and

$$u(0,x) =: \frac{1}{\left(T'\right)^\beta} w_{a,00} \qquad v\left(0,x\right) =: \frac{1}{\left(T'\right)^\beta} w\left(s_0,y\right) =: \frac{1}{\left(T'\right)^\beta} w_{a,0}$$

where $s_0 := -\log T'$ and more particularly $w_a = w_{T,a}$. We then see that $w(s) = w_{T',a}(s)$ satisfies for all $s \geq -\log T'$ (and $s < -\log\left(T'-T\right)$ if $T' > T$) and all $y \in \mathbb{R}^N$

$$w_s + \beta w + \frac{y}{2}\nabla w - \int_{s_0}^s h\left(s-\tau\right)\Delta w\left(\tau\right) d\tau - h\left(s-s_0\right)\Delta w_{00} = \left|w\right|^{p-1} w \tag{7}$$

where $h(s) := e^{-(\beta+1)s}$, or equivalently

$$\begin{aligned} g(s)\, w_s + \beta g(s)\, w + g(s)\tfrac{y}{2}\nabla w - \textstyle\int_{s_0}^{s} g(\tau)\,\Delta w d\tau - g(s_0)\,\Delta w_{00} \\ = g(s)\,|w|^{p-1}\, w \end{aligned} \tag{8}$$

with $g(s) = e^{(\beta+1)s}$.

In the new set of variables (s, y), the behavior of u_t as $t \uparrow T$ is equivalent to the behavior of w as $s \to \infty$.

We are now able to state the following estimate on the function w_a:

Theorem 1.1 ***(Bounds on*** w***)***
Assume $1 < p \leq 1 + \frac{2}{N-1}$ *and* $p < 3$ *if* $N = 1$, $p \neq 3$ *if* $N = 2$. *If* u *is a blowing-up solution at time* $T > 0$ *of equation (1) and* w_a *is defined as in (6) and satisfies:*

$$\begin{aligned} E(w)(s_0) \;:=\; & \frac{\beta}{2}\int_B \rho^{\alpha} w_{a,0}^2 dy - \frac{1}{p+1}\int_B \rho^{\alpha}\,|w_{a,0}|^{p+1}\, dy \\ & + \frac{1}{2}\left\{\int_B \rho^{\alpha}\,|\nabla w_{a,00} + \nabla w_{a,0}|^2\, dy - \int_B \rho^{\alpha}\,|\nabla w_{a,0}|^2\, dy\right\} \\ & + \alpha\left\{\int_B \rho^{\alpha-1}\,[y\nabla w_{a,00} - w_{a,0}]^2\, dy - \int_B \rho^{\alpha-1}\,[w_{a,0}]^2\, dy\right\} \geq 0, \end{aligned} \tag{9}$$

then there exists a constant $K > 0$ *such that*

$$\sup_{\substack{s > s_0 \\ a \in \mathbb{R}^N}} \left[\left\|\int_{s_0}^{s} w_a\left(s', y\right) ds'\right\|_{H^1(B)}^2 + \|w_a\,(s, y)\|_{L^2(B)}^2\right] < K$$

where B *denotes the unit ball of* $\mathbb{R}^N$*.*

This can be translated in terms of u:

Proposition 1.1 ***(Bounds on blow-up solutions)***
Assume $1 < p \leq 1 + \frac{2}{N-1}$ *and* $p < 3$ *if* $N = 1$, $p \neq 3$ *if* $N = 2$. *If* u *is a blowing-up solution at time* $T > 0$ *of equation (1) and* w_a *is defined as in (6) satisfying (9), then there exists positive constants* C_1 *and* C_2 *such that for all* $t \in (0, T)$ *and all* $a \in \mathbb{R}^N$*:*

$$\|u\|_{H^1\left(B_{a,\sqrt{T-t}}\right)}^2 + \|u_t\,(t)\|_{L^2\left(B_{a,\sqrt{T-t}}\right)}^2 \leq \frac{C_1}{(T-t)^{2\beta - \frac{N}{2}}},$$

which implies that

$$\sup_{\substack{0 \leq t < T \\ a \in \mathbb{R}^N}} (T-t)^{\beta}\left[\|u_t\|_{L^2(B_a)}^2 + \|u\|_{H^1(B_a)}\right] \leq C_2,$$

where B_a *is the unit ball centered at* a *and* $B_{a,\sqrt{T-t}}$ *is the ball of center* a *and radius* $\sqrt{T-t}$*.*

Remark 1.1 *In the case $N = 1$, and by Duhamel's formula (see [1]), we are able to prove that these bounds are optimal.*

In section 2, we define a decreasing in time weighted energy, and derive classical blow-up results for equations (1) and (8).

Section 3 is devoted to the proofs of Theorem 1.1 and Proposition 1.1.

2 A blow-up result for equation (8)

2.1 The associated energy

In this subsection we define a weighted energy associated to the equation (8). Denote by α any number satisfying $\alpha > \max\{\beta(\beta+1)/2, 2\}$, and $\rho(y) := 1 - |y|^2$. Define the energy E associated to (8) as follows:

$$\begin{aligned}
E(s) \;=\; & \frac{\beta}{2}\int_B g(s)\rho^\alpha w^2 dy - \frac{1}{p+1}\int_B g(s)\rho^\alpha |w|^{p+1} dy \\
& -\frac{1}{2}\int_{s_0}^{s}\int_B g(\tau)\rho^\alpha \left\{ |\nabla w(\tau) - \nabla w(s)|^2 - |\nabla w(s)|^2 + 2\,|\nabla w(\tau)|^2 \right\} dy d\tau \\
& +\alpha \int_{s_0}^{s}\int_B g(\tau)\rho^{\alpha-2}\left\{[w(\tau)-w(s)]^2 - w^2(s)\right\}\left[(2(\alpha-1)+N)|y|^2 - N\right] dy d\tau \\
& -\alpha \int_{s_0}^{s}\int_B g(\tau)\rho^{\alpha-1}\left\{[w(\tau) - y\nabla w(s)]^2 - [y\nabla w(s)]^2\right\} dy d\tau \\
& +\frac{g(s_0)}{2}\int_B \rho^\alpha \left[|\nabla w_{00} + \nabla w|^2 dy - |\nabla w|^2\right] dy \\
& +\alpha g(s_0)\int_B \rho^{\alpha-1}\left\{[y\nabla w_{00} - w]^2 - w^2\right\} dy.
\end{aligned}$$

We start with

Lemma 2.1 *The energy $s \mapsto E(s)$ is a decreasing function for $s \geq s_0$. Moreover, we have*

$$\begin{aligned}
E(s+1) - E(s) \;=\; & -\frac{\beta+1}{p+1}\int_s^{s+1} g(s')\int_B \rho^\alpha |w(s')|^{p+1} dy ds' \\
& -(1-|y|^2/8)\int_s^{s+1} g(s')\int_B \rho^\alpha [w_{s'}(s')]^2 dy ds' \\
& -(\alpha - \beta(\beta+1)/2)\int_s^{s+1} g(s')\int_B \rho^\alpha w^2(s') dy ds' \\
& -\alpha \int_s^{s+1} g(s')\int_B \rho^{\alpha-1}|y w(s')|^2 dy ds' \\
& -\frac{1}{2}\int_s^{s+1} g(s')\int_B \rho^\alpha \left|\nabla w(s') + \frac{y}{2} w_s(s')\right|^2 dy ds'.
\end{aligned} \tag{10}$$

Proof: In order to calculate the derivative of E, multiply the equation (8) by $\rho^\alpha w_s$ and integrate over B, then we get

$$\begin{aligned}
&\frac{1}{p+1}\frac{d}{ds}\int_B g(s)\rho^\alpha |w|^{p+1}dy - \frac{\beta+1}{p+1}\int_B g(s)\rho^\alpha |w|^{p+1}dy \qquad (11)\\
&= \int_B g(s)\rho^\alpha w_s^2 dy + \frac{\beta}{2}\frac{d}{ds}\int_B g(s)\rho^\alpha w^2 dy - \frac{\beta(\beta+1)}{2}\int_B g(s)\rho^\alpha w^2 dy\\
&+ \int_B g(s)\frac{y}{2}\rho^\alpha \nabla w w_s dy - \int_B\int_{s_0}^s g(\tau)\Delta w(\tau)\rho^\alpha w_s(s)d\tau dy - g(s_0)\int_B \rho^\alpha w_s \Delta w_{00} dy,
\end{aligned}$$

which is equivalent to

$$\begin{aligned}
&\frac{\beta}{2}\frac{d}{ds}\int_B g(s)\rho^\alpha w^2 dy - \frac{1}{p+1}\frac{d}{ds}\int_B g(s)\rho^\alpha |w|^{p+1}dy + \int_B g(s)\frac{y}{2}\rho^\alpha \nabla w w_s dy \qquad (12)\\
&+ \int_B\int_{s_0}^s g(\tau)\rho^\alpha \nabla w(\tau)\nabla w_s(s)d\tau dy - 2\alpha\int_B\int_{s_0}^s g(\tau)\rho^{\alpha-1}y\nabla w(\tau)w_s(s)d\tau dy\\
&+ g(s_0)\int_B \rho^\alpha \nabla w_{00}\nabla w_s dy - 2\alpha g(s_0)\int_B \rho^{\alpha-1}y\nabla w_{00}w_s dy\\
&= -\frac{(\beta+1)}{p+1}\int_B g(s)\rho^\alpha |w|^{p+1}dy - \int_B g(s)\rho^\alpha w_s^2 dy + \frac{\beta}{2}(\beta+1)\int_B g(s)\rho^\alpha w^2 dy,
\end{aligned}$$

that we rewrite as follows:

$$\begin{aligned}
&\frac{\beta}{2}\frac{d}{ds}\int_B g(s)\rho^\alpha w^2 dy - \frac{1}{p+1}\frac{d}{ds}\int_B g(s)\rho^\alpha |w|^{p+1}dy + I_0 + I_1 + I_2 + I_3 + I_4 \quad (13)\\
&= -\frac{(\beta+1)}{p+1}\int_B g(s)\rho^\alpha |w|^{p+1}dy - \int_B g(s)\rho^\alpha w_s^2 dy + \frac{\beta}{2}(\beta+1)\int_B g(s)\rho^\alpha w^2 dy.
\end{aligned}$$

Now, we rewrite $I_0, \cdots, I_4$ one by one as follows:

$$\begin{aligned}
I_0 &:= \int_B g(s)\frac{y}{2}\rho^\alpha \nabla w w_s dy\\
&= -\frac{d}{ds}\int_B\int_{s_0}^s g(\tau)\rho^\alpha |\nabla w(\tau)|^2 d\tau dy + \frac{g(s)}{2}\int_B \rho^\alpha |\nabla w|^2 dy\\
&\quad + \frac{1}{2}\int_B g(s)\rho^\alpha |\nabla w + \frac{y}{2}w_s|^2 dy - \frac{1}{2}\int_B g(s)\rho^\alpha |\frac{y}{2}w_s|^2 dy,
\end{aligned}$$

$$\begin{aligned}
I_1 &:= \int_B\int_{s_0}^s g(\tau)\rho^\alpha \nabla w(\tau)\nabla w_s(s)d\tau dy\\
&= -\frac{1}{2}\frac{d}{ds}\int_B\int_{s_0}^s \rho^\alpha g(\tau)|\nabla w(\tau) - \nabla w(s)|^2 d\tau dy\\
&\quad + \frac{1}{2}\frac{d}{ds}\int_{s_0}^s g(\tau)d\tau\int_B \rho^\alpha |\nabla w(s)|^2 dy - \frac{1}{2}g(s)\int_B \rho^\alpha |\nabla w|^2 dy,
\end{aligned}$$

$$
\begin{aligned}
I_3 &:= g(s_0)\int_B \rho^\alpha \nabla w_{00} \nabla w_s\, dy \\
&= \frac{1}{2} g(s_0)\frac{d}{ds}\left\{\int_B \rho^\alpha |\nabla w_{00}+\nabla w|^2 dy - \int_B \rho^\alpha |\nabla w|^2 dy\right\},
\end{aligned}
$$

$$
\begin{aligned}
I_4 &:= -2\alpha g(s_0)\int_B \rho^{\alpha-1} y\nabla w_{00} w_s\, dy \\
&= \alpha g(s_0)\frac{d}{ds}\left\{\int_B \rho^{\alpha-1}[y\nabla w_{00}-w]^2 dy - \int_B \rho^{\alpha-1} w^2\, dy\right\}.
\end{aligned}
$$

Remainder I_2:

$$
\begin{aligned}
I_2 &:= -2\alpha \int_B \int_{s_0}^s g(\tau)\rho^{\alpha-1}\nabla w(\tau) y w_s(s) d\tau dy \\
&= 2\alpha \int_B \int_{s_0}^s g(\tau) w(\tau)\nabla(y\rho^{\alpha-1} w_s(s)) d\tau dy \\
&= 2\alpha \int_B \int_{s_0}^s g(\tau) w(\tau) N \rho^{\alpha-1} w_s(s) d\tau dy \\
&\quad -4\alpha(\alpha-1)\int_B \int_{s_0}^s g(\tau) w(\tau)|y|^2 \rho^{\alpha-2} w_s(s) d\tau dy \\
&\quad +2\alpha \int_B \int_{s_0}^s g(\tau) w(\tau) y\rho^{\alpha-1}\nabla w_s(s) d\tau dy \\
&= A_1 + A_2 + A_3,
\end{aligned}
$$

with

$$
\begin{aligned}
A_1 &:= 2\alpha N \int_B \int_{s_0}^s g(\tau) w(\tau)\rho^{\alpha-1} w_s(s)\, d\tau dy \\
&= -\alpha N \frac{d}{ds}\int_B \int_{s_0}^s g(\tau)\rho^{\alpha-1}[w(\tau)-w(s)]^2 d\tau dy \\
&\quad +\alpha N \frac{d}{ds}\int_{s_0}^s g(\tau) d\tau \int_B \rho^{\alpha-1} w^2 dy \\
&\quad -\alpha N \int_B g(s)\rho^{\alpha-1} w^2 dy,
\end{aligned}
$$

$$
\begin{aligned}
A_2 &:= -4\alpha(\alpha-1)\int_B \int_{s_0}^s g(\tau) w(\tau)|y|^2\rho^{\alpha-2} w_s(s)\, d\tau dy \\
&= 2\alpha(\alpha-1)\frac{d}{ds}\int_B \int_{s_0}^s g(\tau)|y|^2\rho^{\alpha-2}[w(\tau)-w(s)]^2 d\tau dy \\
&\quad -2\alpha(\alpha-1)\frac{d}{ds}\int_{s_0}^s g(\tau) d\tau \int_B |y|^2\rho^{\alpha-2} w^2\, dy \\
&\quad +2\alpha(\alpha-1)\int_B g(s)|y|^2\rho^{\alpha-2} w^2\, dy,
\end{aligned}
$$

$$
\begin{aligned}
A_3 &:= 2\alpha \int_B \int_{s_0}^s g(\tau) w(\tau) y \rho^{\alpha-1} \nabla w_s(s) d\tau dy \\
&= -\alpha \frac{d}{ds} \int_B \int_{s_0}^s g(\tau) \rho^{\alpha-1} [w(\tau) - y\nabla w(s)]^2 d\tau dy \\
&\quad +\alpha \frac{d}{ds} \int_{s_0}^s g(\tau) d\tau \int_B \rho^{\alpha-1} [y\nabla w(s)]^2 dy \\
&\quad -\alpha \int_B g(s) \rho^{\alpha-1} [y\nabla w(s)]^2 dy \\
&\quad +\alpha \int_B g(s) \rho^{\alpha-1} [w(s) - y\nabla w(s)]^2 dy \\
&= -\alpha \frac{d}{ds} \int_B \int_{s_0}^s g(\tau) \rho^{\alpha-1} [w(\tau) - y\nabla w(s)]^2 d\tau dy \\
&\quad +\alpha \frac{d}{ds} \int_{s_0}^s g(\tau) d\tau \int_B \rho^{\alpha-1} [y\nabla w(s)]^2 dy \\
&\quad +\alpha \int_B g(s) \rho^{\alpha-1} w(s)^2 dy - \alpha \int_B g(s) \rho^{\alpha-1} y \nabla [w(s)^2] dy \\
&= -\alpha \frac{d}{ds} \int_B \int_{s_0}^s g(\tau) \rho^{\alpha-1} [w(\tau) - y\nabla w(s)]^2 d\tau dy \\
&\quad +\alpha \frac{d}{ds} \int_{s_0}^s g(\tau) d\tau \int_B \rho^{\alpha-1} [y\nabla w(s)]^2 dy \\
&\quad +\alpha \int_B g(s) \rho^{\alpha-1} w(s)^2 dy + \alpha \int_B g(s) \nabla [\rho^{\alpha-1} y] w(s)^2 dy \\
&= -\alpha \frac{d}{ds} \int_B \int_{s_0}^s g(\tau) \rho^{\alpha-1} [w(\tau) - y\nabla w(s)]^2 d\tau dy \\
&\quad +\alpha \frac{d}{ds} \int_{s_0}^s g(\tau) d\tau \int_B \rho^{\alpha-1} [y\nabla w(s)]^2 dy \\
&\quad +\alpha \int_B g(s) \rho^{\alpha-1} w(s)^2 dy + \alpha N \int_B g(s) \rho^{\alpha-1} w(s)^2 dy \\
&\quad -2\alpha(\alpha-1) \int_B g(s) \rho^{\alpha-2} |y|^2 w(s)^2 dy.
\end{aligned}
$$

Then

$$
\begin{aligned}
I_2 &= -\alpha N \frac{d}{ds} \int_B \int_{s_0}^s g(\tau) \rho^{\alpha-1} [w(\tau) - w(s)]^2 d\tau dy \\
&\quad +\alpha N \frac{d}{ds} \int_{s_0}^s g(\tau) d\tau \int_B \rho^{\alpha-1} w^2 dy \\
&\quad +2\alpha(\alpha-1) \frac{d}{ds} \int_B \int_{s_0}^s g(\tau) |y|^2 \rho^{\alpha-2} [w(\tau) - w(s)]^2 d\tau dy \\
&\quad -2\alpha(\alpha-1) \frac{d}{ds} \int_{s_0}^s g(\tau) d\tau \int_B |y|^2 \rho^{\alpha-2} w(s)^2 dy
\end{aligned}
$$

$$
\begin{aligned}
&-\alpha\frac{d}{ds}\int_B\int_{s_0}^s g(\tau)\rho^{\alpha-1}[w(\tau)-y\nabla w(s)]^2 d\tau dy\\
&+\alpha\frac{d}{ds}\int_{s_0}^s g(\tau)d\tau\int_B \rho^{\alpha-1}[y\nabla w(s)]^2 dy+\alpha\int_B g(s)\rho^{\alpha-1}w^2 dy.
\end{aligned}
$$

Putting $I_0,\cdots,I_4$ into (13) we finally get

$$
\begin{aligned}
\frac{d}{ds}E(s) \quad = \quad & -\frac{\beta+1}{p+1}\int_B g(t)\rho^\alpha|w|^{p+1}\,dy-[\alpha-\beta(\beta+1)/2]\int_B g(t)\rho^\alpha w^2\,dy\\
&-\alpha\int_B g(t)\rho^{\alpha-1}|yw|^2 dy-(1-|y|^2/8)\int_B g(s)\rho^\alpha w_s^2\,dy\\
&-\frac{1}{2}\int_B g(s)\rho^\alpha\left|yw_s/2+\nabla w\right|^2\,dy,
\end{aligned}
$$

which terminates the proof of the Lemma. □

2.2 Blow-up results

We are now able to state and prove the blow-up results for equations (8) and (1)

Theorem 2.1 *Assume that $1<p\le 1+\frac{2}{N}$ and w is a solution of (8) on B such that $E[w](s_0)<0$, for some $s_0\in\mathbb{R}$, then w blows-up in $H^1(B)\times L^2(B)$ at a time $s^*>s_0$.*

This implies directly the following blow-up result for (5).

Proposition 2.1 *Assume that $1<p\le 1+\frac{2}{N}$ and v is a solution of (5) on B such that $\mathcal{E}_{T,a}[v](t):=E[w_{T,a}](-\log(T-t))<0$ for some $0\le t\le T$, $a\in\mathbb{R}^N$, then v blows-up in finite time $T'<T$.*

Proof of Proposition 2.1. Assume that there exists $T>0$, $0<t_0<T$ and $a\in\mathbb{R}^N$ such that $\mathcal{E}_{T,a}[v](t_0)<0$. Putting $s_0=-\log(T-t_0)$, then $E(w_{T,a})(s_0)<0$. Applying Theorem 2.1, the solution w of (8) blows-up in finite time $s^*<\infty$. As $v(t,x)=\frac{1}{(T-t)^\beta}w(s,y)$, we deduce that v blows-up in finite time T' such that $s^*=-\log(T-t^*)\ge-\log(T-T')$, thus $T'\le T-e^{-s^*}<T$. □

Proof of Theorem 2.1: Arguing by contradiction, assume that there exists a solution $w(s,y)$ of equation (8) on $(s,y)\in[s_0,\infty)\times B$, with $E[w](s_0)<0$. As E is decreasing, $E[w](s_0+1)<0$. We see that $v(t,x)=\frac{1}{(-t)^\beta}w_{0,0}(s,y)$ is a solution of (1) and $w_{0,0}$ is a solution of (8) on $(s,y)\in[s_0,\infty)\times B$ such that $y=\frac{x}{\sqrt{-t}}\in B$, $s=-\ln(-t)$ for all $t<-e^{-s_0}$. Put $t:=(\delta+t')=-e^{-s}$ where δ is the positive constant small enough such that $-\ln[e^{-(s_0+1)}+\delta]\ge s_0$,

then

$$
\begin{aligned}
v(t',x) &= \frac{1}{(-t')^{\beta}} w_{0,0}\left(-\ln(-t'), x/\sqrt{-t'}\right) \qquad (14)\\
&= \frac{1}{(\delta+e^{-s})^{\beta}} w_{0,0}\left(-\ln(\delta+e^{-s}), x/\sqrt{\delta+e^{-s}}\right)\\
&= \frac{e^{-\beta s}}{(\delta e^{s}+1)^{\beta}} w_{0,0}\left(-\ln(\delta+e^{-s}), e^{\frac{s}{2}}x/\sqrt{\delta e^{s}+1}\right)
\end{aligned}
$$

is also a solution of (1). We see that

$$
y' = \frac{x}{\sqrt{-t'}} = \frac{x}{\sqrt{\delta+e^{-s}}} = \frac{e^{\frac{s}{2}}x}{\sqrt{\delta e^{s}+1}} = \frac{y}{\sqrt{\delta e^{s}+1}},
$$

then

$$
\begin{aligned}
v(-\delta+t,x) &= \frac{e^{\beta s}}{(\delta e^{s}+1)^{\beta}} w_{0,0}\left(-\ln(\delta+e^{-s}), y/\sqrt{\delta e^{s}+1}\right)\\
&= e^{\beta s}\widetilde{w}(s,y)
\end{aligned}
$$

$$
\widetilde{w}(s,y) = \frac{1}{(e^{s}\delta+1)^{\beta}} w_{0,0}(-\ln(e^{-s}+\delta), y/\sqrt{(e^{s}\delta+1)})
$$

is a solution of (8) on $[s_0+1,\infty)\times B$. Using the inequality $(a-b)^2 \leq 2(a^2+b^2)$, Poincaré's inequality and using the fact that $\rho^{\alpha} \leq \rho^{\alpha-1} \leq \rho^{\alpha-2}$ we finally get

$$
\begin{aligned}
E(s) &\geq -\frac{1}{p+1}\int_B g(s)\rho^{\alpha}|\widetilde{w}|^{p+1}dy - \int_B\int_{s_0}^{s} g(\tau)\rho^{\alpha}|\nabla\widetilde{w}(\tau)|^2 d\tau dy\\
&\quad -\frac{1}{2}\int_B\int_{s_0}^{s} \rho^{\alpha}g(\tau)|\nabla\widetilde{w}(\tau)-\nabla\widetilde{w}(s)|^2 d\tau dy\\
&\quad -\alpha N\int_B\int_{s_0}^{s} g(\tau)\rho^{\alpha-1}[\widetilde{w}(\tau)-\widetilde{w}(s)]^2 d\tau dy\\
&\quad -2\alpha(\alpha-1)\int_{s_0}^{s} g(\tau)d\tau\int_B \rho^{\alpha-2}|y\widetilde{w}|^2 dy\\
&\quad -\alpha\int_B\int_{s_0}^{s} g(\tau)\rho^{\alpha-1}(\widetilde{w}(\tau)-y\nabla\widetilde{w}(s))^2 d\tau dy\\
&\quad -\frac{g(s_0)}{2}\int_B \rho^{\alpha}|\nabla\widetilde{w}|^2 dy - \alpha g(s_0)\int_B \rho^{\alpha-1}\widetilde{w}^2 dy\\
E(s) &\geq -Cg(s)\sup_{s\geq s_0}\left\{\int_B \rho^{\alpha-2}|\nabla\widetilde{w}(s)|^2 dy - \frac{1}{p+1}g(s)\int_B \rho^{\alpha}|\widetilde{w}|^{p+1}dy\right\}.
\end{aligned}
$$

By the definition of $\widetilde{w}$ we then obtain

$$
E(s) \geq -C\frac{g(s)}{(\delta e^{s}+1)^{2\beta}}\sup_{s\geq s_0}\int_B \rho^{\alpha-2}\left|\nabla(w(-\ln(\delta+e^{-s}), \frac{y}{\sqrt{\delta e^{s}+1}}))\right|^2 dy
$$

$$
\begin{aligned}
& -\frac{1}{p+1}\frac{g(s)}{(\delta e^s+1)^{2\beta+1}}\int_B \rho^\alpha |w|^{p+1}dy \\
\geq \quad & -C\frac{g(s)}{(\delta e^s+1)^{2\beta+1}}\sup_{s\geq s_0}\int_B \rho^{\alpha-2}\left|\nabla w(-\ln(\delta+e^{-s}),\frac{y}{\sqrt{\delta e^s+1}})\right|^2 dy \\
& -\frac{1}{p+1}\frac{g(s)}{(\delta e^s+1)^{2\beta+1}}\int_B \rho^\alpha \left|w(-\ln(\delta+e^{-s}),\frac{y}{\sqrt{\delta e^s+1}})\right|^{p+1} dy.
\end{aligned}
$$

Now, since $lim_\infty -\log(\delta+e^{-s}) = -\log\delta$, by a continuity argument $w(-\ln(e^{-s}+\delta))$ remains bounded in $H^1(B)$, hence in $L^{p+1}(B)$ by Sobolev's embedding. Using Poincaré's inequality, we get

$$
\begin{aligned}
E(s) &\geq -C\frac{g(s)}{(\delta e^s+1)^{2\beta+1-\frac{N}{2}}} \\
E(s) &\geq -\frac{C}{(1+\delta e^s)^{\beta-\frac{N}{2}}}.
\end{aligned}
$$

Thus,

$$\liminf_{s\to\infty} E(s)\geq 0$$

which is a contradiction. □

3 Uniform bounds on w: proof of Theorem 1.1

3.1 Proof of Theorem 1.1

This section is devoted to prove Theorem 1.1. We start by giving the proof of Proposition 1.1:

Proof of Proposition 1.1. We have

$$\|w_a(s,y)\|^2_{L^2(B)} = (T-t)^{2\beta}\int_B u_t^2(t,x)dy = (T-t)^{2\beta-\frac{N}{2}}\int_{B_{a,\sqrt{T-t}}} u_t^2(t,x)dx.$$

As, by Theorem 1.1, $\|w_a(s,y)\|^2_{L^2(B)}$ is bounded, then

$$\|u_t(t)\|^2_{L^2(B_{a,\sqrt{T-t}})}\leq \frac{C_1}{(T-t)^{2\beta-\frac{N}{2}}},$$

and

$$
\begin{aligned}
& \left\|\int_{s_0}^s w_a(s',y)ds'\right\|^2_{H^1(B)} \\
&= \int_B\left|\int_{s_0}^s e^{-\beta s'}u_t(t,x)ds'\right|^2 dy + \int_B\left|\int_{s_0}^s e^{-\beta s'}\nabla_y u_t(t,x)ds'\right|^2 dy \\
&\geq (T-t)^{2\beta-\frac{N}{2}}\int_{B'}\left[\left|\int_0^t u_t(\tau,x)(T-\tau)^{-1}d\tau\right|^2 + \left|\int_0^t \nabla u_t(t,x)(T-\tau)^{-\frac{1}{2}}d\tau\right|^2\right]dx
\end{aligned}
$$

$$\geq c(T-t)^{2\beta-\frac{N}{2}} \int_{B'} \left\{ \left[\int_0^t u_t(\tau,x)d\tau \right]^2 + \left| \int_0^t \nabla u_t(t,x)d\tau \right|^2 \right\} dx,$$

where $c = \min[T^{-2}, T^{-1}]$ and $B' = B_{a,\sqrt{T-t}}$.
As $\|w_a(s,y)\|^2_{H^1(B)}$ is bounded, then

$$\|u(t)\|^2_{H^1(B_{a,\sqrt{T-t}})} \leq \frac{C_1}{(T-t)^{2\beta-\frac{N}{2}}}.$$

□

In order to prove Theorem 1.1, we first give a local bounds on w_s, w and ∇w in the space $L^2_{t,x}$ (space and time integration) using the Lyapounov functional E. Finally, we deduce a local L^2-estimate on w. This will be done in four steps. We will state them in propositions and lemmas and then prove the Theorem. The proof of the propositions and lemmas will be given at the end of the section.

First step: Uniform L^2-estimates on w_a, $w_{a\tau}$ and ∇w.

Proposition 3.1 *There exists a constant $M_0 > 0$ such that, for all $s \geq s_0$,*

$$\begin{aligned} L(s) \quad := \quad & \int_s^{s+1} g(s') \int_{B/2} |w_a(s')|^{p+1} dyds' + \int_s^{s+1} g(s') \int_{B/2} [(w_a)_{s'}(s')]^2 dyds' \qquad (15) \\ & + \int_s^{s+1} g(s') \int_{B/2} w_a^2(s') dyds' + \int_s^{s+1} g(s') \int_{B/2} |\nabla w_a(s') + \frac{y}{2}(w_a)_s(s')|^2 dyds' \\ \leq \quad & M_0. \end{aligned}$$

Second step: Uniform L^2_x-estimates on w.

Proposition 3.2 *There exists a constant $M > 0$ such that*

$$\sup_{a\in\mathbb{R}^n,\, s\geq s_0} \int_B w_a(s,y)|^2 dy \leq M. \qquad (16)$$

Third step: Uniform $L^2_{t,x}$-estimates on $\int_{s_0}^s h(s-s')\nabla w_a(s',y)\,ds'$.

Proposition 3.3 *There exists a constant $M > 0$ such that*

$$\sup_{a\in\mathbb{R}^N,\, \alpha\geq s_0} \int_\alpha^{\alpha+1} \int_B \left| \int_{s_0}^s h(s-s')\nabla w_a(s',y)ds' \right|^2 dyds \leq M.$$

Fourth step: Uniform L^2_x-estimates on $\int_{s_0}^s w(s')\,ds'$.

Proposition 3.4 *There exists a constant $M > 0$ such that*

$$\sup_{a\in\mathbb{R}^n} \int_B \left[\int_{s_0}^s w_a(s',y)ds' \right]^2 dy \leq M.$$

We will need the following Lemma from [4]:

Lemma 3.1 *[see [4]] Let u be the solution of*

$$\begin{cases} u_{tt} - \Delta u = f, \quad on\ [0, T_0], \\ u(x,0) = u_0 \in H^1_{loc,\,u}(\mathbb{R}^N), \\ u_t(x,0) = u_1 \ \in L^2_{loc,\,u}(\mathbb{R}^N), \end{cases}$$

Then there exists $M > 0$ such that, $\forall t \in [0,\ T_0]$, we have

$$\|u(t)\|^2_{H^1(\mathbb{R}^N)} + \|u_t(t)\|^2_{L^2(\mathbb{R}^N)} \leq M \left[\|u_0\|^2_{H^1(\mathbb{R}^N)} + \|u_1\|^2_{L^2(\mathbb{R}^N)} + \|f\|^2_{L^1_t([0,t];\ L^2_x(\mathbb{R}^N))}\right].$$

End of the proof of Theorem 1.1:

Let $a \in \mathbb{R}^N$, $s_1 \geq s_0 + 1$ and define the self similar transformation as follows

$$\forall s \in [s_1 - 1,\ s_1], \forall y \in B \begin{cases} s - s_0 = (s_1 - s_0)\ln(1-t), \\ \\ y = \frac{x}{\sqrt{1-t}} \\ \\ W_t(t,x) = (1-\tau)^{-\beta} w_a(s,y). \end{cases} \tag{17}$$

Note that $(t,x) \in [1-e,\ 0] \times B_t$, $B_t = B(0, \sqrt{1-t})$. From the fact that w_a is a solution of (8), then W is a solution of (5) on $[1-e, 0] \times B_t$. Observe that $\forall s \in [s_1 - 1, s_1]$, we have, for all $(t,x) \in [1-e,\ 0] \times B_t$

$$\begin{cases} e^{\frac{s-s_0}{s_1-s_0}} = 1 - t, \\ \\ x = y e^{\frac{s-s_0}{2(s_1-s_0)}}, \\ \\ W_t(t,x) = e^{-\beta(\frac{s-s_0}{s_1-s_0})} w_a(s,y). \end{cases}$$

Applying Lemma 3.1 on the interval $[t, t_0 = 1 - \exp(1 - 1/(s_1 - s_0))]$ we have, for all $t \in [1-e, t_0]$,

$$\begin{aligned} \|W(t_0)\|^2_{H^1(B_{t_0})} + \|W_t(t_0)\|^2_{L^2(B_{t_0})} \quad &\leq \quad M \Big(\|W_t\|^2_{L^2(B_t)} + \|W(t)\|^2_{H^1(B_t)} \\ &\qquad + \|W_t^p\|^2_{(L^1[[t,t_0];\ L^2(B_t)])}\Big). \end{aligned} \tag{18}$$

Now, we will use propositions 3.1, 3.2, 3.3 and 3.4 to estimate the right-hand side of the above inequality. We have that, for all $t \in [1-e,\ t_0]$,

$$\|W_t(t)\|^2_{L^2(B_t)} = \int_B e^{\left(\frac{s-s_0}{s_1-s_0}\right)(N/2-2\beta)} |w(s,y)|^2 dy \leq M. \tag{19}$$

Now,

$$
\begin{aligned}
\|W(t)\|^2_{H^1(B_t)} &= \int_{B_t}\left|W(0,x)-\int_t^0(1-\tau)^{-\beta}w_a(s',y(x))d\tau\right|^2 dx \\
&\quad+\int_{B_t}\left|\nabla W(0,x)-\int_t^0(1-\tau)^{-\beta}\nabla w_a(s',y)d\tau\right|^2 dx \\
&\le 2\int_{B_t}|W(0,x)|^2dx+2\int_{B_t}\left|\int_t^0(1-\tau)^{-\beta}w_a(s',y)d\tau\right|^2 dx \\
&\quad+2\int_{B_t}|\nabla W(0,x)|^2dx+2\int_{B_t}\left|\int_t^0(1-\tau)^{-\beta}\nabla_x w_a(s',y)d\tau\right|^2 dx \\
&\quad\le 2\int_{B_{1-e}}W^2(0,x)dx+2\int_{B_{1-e}}|\nabla W(0,x)|^2dx \\
&\quad+\frac{2}{(s_1-s_0)^2}\int_B e^{\frac{N(s-s_0)}{2(s_1-s_0)}}\left[\int_{s_0}^s e^{\frac{(-\beta+1)(s'-s_0)}{s_1-s_0}}w_a(s',y)ds'\right]^2 dy \\
&\quad+\frac{2}{(s_1-s_0)^2}\int_B e^{\frac{N(s-s_0)}{2(s_1-s_0)}}\left|\int_{s_0}^s e^{\frac{(-\beta+1/2)(s'-s_0)}{s_1-s_0}}\nabla w_a(s',y)ds'\right|^2 dy.
\end{aligned}
$$

Since $s\in[s_1-1,\ s_1]$ and $s_1\ge s_0+1$ we have $0\le \frac{s'-s_0}{s_1-s_0}\le 1$. Therefore,

$$
\begin{aligned}
\|W(t)\|^2_{H^1(B_t)} &\le 2\int_{B_{1-e}}W^2(0,x)dx+2\int_{B_{1-e}}|\nabla W(0,x)|^2dx \qquad (20)\\
&\quad+2C\int_B\left[\int_{s_0}^s w_a(s',y)ds'\right]^2 dy+2C\int_B\left|\int_{s_0}^s\nabla w_a(s',y)ds'\right|^2 dy \\
&\quad\le CM.
\end{aligned}
$$

Now, for the last term of (18), we have, denoting by $K_t := [t,\ t_0]\times B_t$, $k_0(s') := \frac{1}{s_1-s_0}\exp\left[(N/2-2\beta)\,(s'-s_0)/(s_1-s_0)\right]$ and $k_1(s') := \frac{1}{s_1-s_0}\exp\left[(N/2-2\beta-1)\,(s'-s_0)/(s_1-\right.$

$$
\begin{aligned}
\|W_t\|^2_{H^1(K_t)} &= \int_{s_1-1}^s\int_B k_0(s')w_a^2(s',y)dyds'+\int_{s_1-1}^s\int_B k_0(s')|\nabla_x(w_a(s',y))|^2dyds' \\
&\quad+\int_{s_1-1}^s\int_B k_0(s')[(w_a(s',y))_\tau]^2dyds' \\
&= \int_{s_1-1}^s\int_B k_0(s')w_a^2(s',y)dyds'+\int_{s_1-1}^s\int_B k_1(s')|\nabla_y(w_a(s',y))|^2dyds' \\
&\quad+\int_{s_1-1}^s\int_B k_1(s')\left|\beta w_a(s',y)+w_{a_s}(s',y)+\frac{y}{2}\nabla(w_a(s',y))\right|^2 dyds \\
&\le C\int_{s_1-1}^{s_1}\int_B\left\{[w_a(s',y)]^2+[w_{a_s}(s',y)]^2+|\nabla[w_a(s',y)]|^2\right\}dyds' \\
\|W_t\|^2_{H^1(K_t)} &\le M^{\frac{1}{p}}. \qquad (21)
\end{aligned}
$$

Now, since $p \leq 1 + \frac{2}{N-1}$ then $2p \leq \frac{2(N+1)}{N-1}$ and $K_t = [t,\ t_0] \times B_t \subset \mathbb{R}^{N+1}$, so, by Sobolev's embedding, $L^{2p}(K_t) \hookrightarrow H^1(K_t)$, thus, $\forall t \in [1-e, t_0]$ we have

$$\|W_t^p\|^2_{L^1([t,t_0];L^2(B_t))} \leq (t_0 - t)\|W_t^p\|^2_{L^2(K_t)} \leq C\|W_t(s,y)\|^p_{H^1(K_t)} \leq CM.$$

We insert (19), (20) and (21) into (18) to obtain

$$\|W(t_0)\|^2_{H^1(B_{t_0})} + \|W_t(t_0)\|^2_{L^2(B_{t_0})} \leq K.$$

Where K is a positive constant depending only on p and N. Thus, we deduce that

$$\left\| \int_{s_0}^{s_1-1} w_a(s,y) d\tau \right\|^2_{H^1(B)} + \|w_a(s,y)\|^2_{L^2(B)} \leq K,$$

which terminates the proof of Theorem 1.1. □

3.2 Proof of propositions 3.1, 3.2, 3.3 and 3.4

Proof of Proposition 3.1. Since E is decreasing and bounded by $E_a(0)$ and ρ^α is bounded in the ball $\frac{B}{2}$, we deduce that, for all $s \geq s_0$

$$\begin{aligned} L(s) \quad \leq \quad & C\int_s^{s+1}\int_{\frac{B}{2}} g(s')\rho^\alpha |w(s')|^{p+1} dyds' + \frac{C}{2}\int_s^{s+1}\int_{\frac{B}{2}} g(s')\rho^\alpha w_s^2(s') dyds' \\ & + C(\alpha - \beta(\beta+1)/2)\int_s^{s+1}\int_{\frac{B}{2}} g(s')\rho^\alpha w^2(s') dyds' \\ & + \frac{C}{2}\int_s^{s+1}\int_{\frac{B}{2}} g(s')\rho^\alpha |\nabla w(s') + \frac{y}{2} w_s(s')|^2 dyds' \\ \leq \quad & C[E(s) - E(s+1)] \leq C \sup_{a\in\mathbb{R}^N} E_a(0) = M_0. \end{aligned}$$

□

Proof of Proposition 3.2. By contradiction, suppose that there exists a sequence $s_n \to \infty$ and (a_n) such that

$$\int_B |w_{a_n}(s_n, y)|^2 dy \longrightarrow \infty.$$

For $a \in \mathbb{R}^N$, set

$$v_{n,a}(\tau, z) := \lambda_n^\beta w_a(s_n + \lambda_n \tau,\ \sqrt{\lambda_n} z) \tag{22}$$

and more particularly write $v_n := v_{n,a_n}$, where $\lambda_n > 0$, $a \in \mathbb{R}^N$ are chosen in such a way that

$$\frac{1}{2} \leq \|v_n(0,z)\|^2 \leq \sup_{a_0\in\mathbb{R}^N} \|v_{n,a_0}(0,z)\|^2 = 1. \tag{23}$$

This choice is possible because for all $0 < \lambda \leq 1$,

$$\sup_{a_0 \in \mathbb{R}^N} \|\lambda^\beta w_{a_0}(s_n, \sqrt{\lambda} z)\|^2_{L^2(B)} \leq \lambda^{2\beta - \frac{N}{2}} \sup_{a_0 \in \mathbb{R}^N} \|w_{a_0}(s_n, y)\|^2_{L^2(B)}.$$

and $2\beta - \frac{N}{2} > 0$.
We then have from (22), $\lambda_n \to 0$. We claim now the following estimates, that we prove in Appendix B:

Lemma 3.2 *We have*

(i)

$$\sup_{\tau, \tau' \in [0,1]} \|v_{n,a_0}(\tau) - v_{n,a_0}(\tau')\|^2_{L^2(2B)} \leq \int_0^1 \|(v_{n,a_0})_\tau\|^2_{L^2(2B)} \leq M_0 \lambda^{2\beta - (\frac{N}{2} - 1)}.$$

(ii)

$$\int_0^1 \|\nabla v_{n,a_0}\|^2_{L^2(2B)} d\tau \leq M_0 \lambda_n^{2\beta - \frac{N}{2}}.$$

(iii)

$$\int_0^1 \|v_{n,a_0}\|^2_{L^2(2B)} \leq M_0 \lambda_n^{2\beta - (\frac{N}{2} + 1)}.$$

(iv)

$$\int_0^1 \int_{2B} |v_{n,a_0}|^{p+1} dz d\tau \leq M_0 \lambda_n^{2\beta - \frac{N}{2}}.$$

Back to the proof of Proposition 3.2. Two cases may then occur, depending on whether

$$\lambda_n \sup_{a \in \mathbb{R}^n} \int_{\frac{1}{3}}^{\frac{2}{3}} \int_{2B} \left| \int_{\tau_{n_0}}^{\tau} h((\tau - \tau_1)\lambda_n) |\nabla v_n(\tau_1, z)| d\tau_1 \right|^2 dz d\tau$$

is bounded or not with $\tau_{n_0} = \frac{s_0 - s_n}{\lambda_n}$.
Case A: Up to a subsequence, there exists z_{n_0} such

$$\lambda_n \int_{\frac{1}{3}}^{\frac{2}{3}} \int_{2B} \left| \int_{\tau_{n_0}}^{\tau} h((\tau - \tau_1)\lambda_n) |\nabla v_{n, z_{n_0}}(\tau_1, z)| d\tau_1 \right|^2 dz d\tau \to \infty.$$

We will obtain a contradiction from the variation in some scale of the local L^2-norm.

Let $\varphi_0 \in C^\infty(\mathbb{R})$ be a nonnegative function whose support $\subset (-1,1)$, $\varphi_0 \geq 1$ on $[-\frac{2}{3}, \frac{2}{3}]$ and $\int_{-1}^1 \varphi_0 = 1$.
We use again space time rescaling, that is to say we define

$$\begin{aligned}
\widetilde{v}_{n,a_0,\tau_0}(\tau,z) &= (\widetilde{\lambda}_n/\lambda_n)^\beta v_{n,a_0}\left(\tau_0 + \frac{\widetilde{\lambda}_n}{\lambda_n}\tau, z\sqrt{\widetilde{\lambda}_n/\lambda_n}\right) \\
&= (\widetilde{\lambda}_n)^\beta w_{a_0}(s_n + \lambda_n\tau_0 + \widetilde{\lambda}_n\tau, z\sqrt{\widetilde{\lambda}_n}),
\end{aligned}$$

and more particularly

$$\widetilde{v}_n = \widetilde{v}_{n,b_n,\tau_n}(\tau,z) = (\widetilde{\lambda}_n/\lambda_n)^\beta v_n\left(\tau_n + \frac{\widetilde{\lambda}_n}{\lambda_n}\tau, \frac{(b_n-a_n)e^{\frac{s_n+\lambda_n\tau_n+\widetilde{\lambda}_n\tau}{2}}}{\sqrt{\lambda_n}} + z\sqrt{\widetilde{\lambda}_n/\lambda_n}\right),$$

where $0 < \widetilde{\lambda}_n \leq \lambda_n/3$, $\tau_n \in [\frac{1}{3}, \frac{2}{3}]$ and $b_n \in \mathbb{R}^N$ are chosen so that $\widetilde{v}_n$ satisfies:

$$\begin{aligned}
\frac{1}{2} &\leq \widetilde{\lambda}_n \int_{-1}^1 \varphi_0(\tau)\int_B \left|\int_{\tau_{n_0}}^\tau h(\widetilde{\lambda}_n(\tau-\tau_1))\nabla\widetilde{v}_n(\tau_1,z)d\tau_1\right|^2 dzd\tau \qquad (24)\\
&\leq \sup_{\substack{a_0\in\mathbb{R}^N\\ \tau_0\in[\frac{1}{3},\frac{2}{3}]}} \widetilde{\lambda}_n \int_{-1}^1 \varphi_0(\tau)\int_B \left|\int_{\tau_{n_0}}^\tau h(\widetilde{\lambda}_n(\tau-\tau_1))\nabla\widetilde{v}_{n,a_0,\tau_0}(\tau_1,z)d\tau_1\right|^2 dzd\tau = 1.
\end{aligned}$$

where $s_0 = s_n + \lambda_n\tau_0 + \widetilde{\lambda}_n\tau_{n_0}$. This choice is possible since for all $0 < \widetilde{\lambda} \leq \frac{\widetilde{\lambda}_n}{3}$, we have

$$\begin{aligned}
&\sup_{\substack{a_0\in\mathbb{R}^N\\ \tau_0\in[\frac{1}{3},\frac{2}{3}]}} \widetilde{\lambda}\left(\widetilde{\lambda}/\lambda_n\right)^{2\beta} \\
&\times \int_{-1}^1\int_B \varphi_0(\tau)\left|\int_{\tau_0}^\tau h(\widetilde{\lambda}(\tau-\tau_1))\nabla\left[v_{n,a_0}\left(\tau_0+\tau_1\widetilde{\lambda}/\lambda_n, z\sqrt{\widetilde{\lambda}/\lambda_n}\right)\right]d\tau_1\right|^2 dzd\tau \\
\leq\ & C\|\varphi_0\|_\infty \widetilde{\lambda}\left(\widetilde{\lambda}/\lambda_n\right)^{2\beta+1-\frac{N}{2}-2} \\
&\times \sup_{\substack{a_0\in\mathbb{R}^N\\ \tau_0\in[\frac{1}{3},\frac{2}{3}]}} \int_{-1}^1\int_B \left|\int_{\tau_0+\frac{\widetilde{\lambda}}{\lambda_n}\tau_{n_0}}^{\tau_0+\frac{\widetilde{\lambda}}{\lambda_n}\tau} h\left(\widetilde{\lambda}\tau - \lambda_n(s'-\tau_0)\right)\nabla v_{n,a_0}(s',y)ds'\right|^2 dyd\tau \\
\leq\ & C'\|\varphi_0\|_\infty\left(\widetilde{\lambda}/\lambda_n\right)^{2\beta-\frac{N}{2}} \\
&\times \sup_{\substack{a_0\in\mathbb{R}^N\\ \tau_0\in[\frac{1}{3},\frac{2}{3}]}} \int_{\tau_0-1}^{\tau_0+1}\int_B \left|\int_{\tau_0+\frac{\widetilde{\lambda}}{\lambda_n}\tau_{n_0}}^s h\left(\widetilde{\lambda}(s-s')\right)\nabla v_{n,a_0}(s',y)ds'\right|^2 dyds.
\end{aligned}$$

Now, we need the following estimate, that we prove also in Appendix B:

Lemma 3.3 *We have, for any $\tau \in [-1,1]$*

(a)

$$\frac{\widetilde{\lambda}_n}{\lambda_n} + \int_{-1}^{1}\int_{2B} \widetilde{v}_{n\tau}^2 dz d\tau + \int_{-1}^{1}\int_{2B} |\nabla \widetilde{v}_n|^2\, dz d\tau + \sup_{\tau\in[-1,1]} \int_{2B} \widetilde{v}_n^2 dz + \int_{-1}^{1}\int_{2B} |\widetilde{v}_n|^{p+1} dz d\tau \to 0,$$

(b)

$$\widetilde{\lambda}_n \int_{-1}^{1}\int_{2B} \varphi_0(\tau) \left| \int_{\tau_{n_0}}^{\tau} h((\tau-\tau_1)\widetilde{\lambda}_n)\nabla \widetilde{v}_n(\tau_1,z)d\tau_1 \right|^2 dz d\tau \le c,$$

(c)

$$\lambda_n^2 \int_B \left[\int_{\tau_{n_0}=\frac{s_0-s_n}{\lambda_n}}^{\tau} v_n(\tau,z)d\tau \right]^2 dz \le c.$$

Now, note that $\widetilde{v}_n$ is a solution of

$$(\widetilde{v}_n)_\tau + \widetilde{\lambda}_n\beta\widetilde{v}_n + \widetilde{\lambda}_n \frac{z_n}{2}\nabla\widetilde{v}_n - \widetilde{\lambda}_n \int_{\tau_{n_0}}^{\tau} h(\widetilde{\lambda}_n(\tau-\tau_1))\Delta\widetilde{v}_n d\tau_1 - h(s_n+\lambda_n\tau_n+\widetilde{\lambda}_n\tau - s_0)\Delta\widetilde{v}_{n_{00}} = |\widetilde{v}_n|^{p-1}\widetilde{v}_n. \quad (25)$$

Here, we denote $h(s_n+\lambda_n\tau_n+\widetilde{\lambda}_n\tau - s_0) = h_{n_0}(\tau)$, $s_0 = s_n+\lambda_n\tau_n + \widetilde{\lambda}_n\tau$ and $\widetilde{v}_{n_{00}} = \widetilde{\lambda}_n^{\beta} w_{00}(\sqrt{\widetilde{\lambda}_n}z)$. Then, we deduce $\|\widetilde{v}_{n_{00}}\|^2_{H^1(2B)} \to 0$ since $w_{00}(\sqrt{\widetilde{\lambda}_n}z) = (T')^{\beta}u(0,x)$ and $u(0,x)\in H^1_{loc,u}(\mathbb{R}^N)$.

Choose now $\psi \in C^\infty(\mathbb{R}^n)$ such that $\psi \ge 0$, $\psi = 1$ on B whose support $\subset 2B$. Multiplying equation (25) by

$$\Phi(\tau,z)\widetilde{v}_n := \int_{-1}^{\tau} \varphi_0(\tau_1)d\tau_1\psi(z)\widetilde{v}_n \quad (26)$$

and integrating over $[-1,1]\times 2B$, we obtain the following equality:

$$\int_{-1}^{1}\int_{2B}[(\widetilde{v}_n)_\tau\Phi\widetilde{v}_n + \widetilde{\lambda}_n\beta\Phi\widetilde{v}_n^2 + \widetilde{\lambda}_n\frac{z_n}{2}\nabla\widetilde{v}_n\Phi\widetilde{v}_n]dzd\tau$$
$$+\widetilde{\lambda}_n \int_{-1}^{1}\int_{2B}\int_{\tau_{n_0}}^{\tau} h(\widetilde{\lambda}_n(\tau-\tau_1))\nabla\widetilde{v}_n d\tau_1[\nabla\Phi\widetilde{v}_n + \Phi\nabla\widetilde{v}_n]dzd\tau$$
$$+\int_{-1}^{1}\int_{2B} h_{n_0}\nabla\widetilde{v}_{n_{00}}[\nabla\Phi\widetilde{v}_n + \Phi\nabla\widetilde{v}_n]dzd\tau = \int_{-1}^{1}\int_{2B}|\widetilde{v}_n|^{p+1}\Phi dzd\tau,$$

which is equivalent to

$$
\begin{aligned}
&\frac{\widetilde{\lambda}_n}{2}\int_{-1}^{1}\int_{2B}\left|\int_{\tau_{n_0}}^{\tau} h(\tau-\tau_1)\nabla\widetilde{v}_n d\tau_1\right|^2 \Phi_\tau dz d\tau \\
&= \int_{-1}^{1}\int_{2B}[(\widetilde{v}_n)_\tau\Phi\widetilde{v}_n+\widetilde{\lambda}_n\beta\Phi\widetilde{v}_n^2+\widetilde{\lambda}_n\frac{z_n}{2}\nabla\widetilde{v}_n\Phi\widetilde{v}_n]dzd\tau \\
&+\widetilde{\lambda}_n\int_{-1}^{1}\int_{2B}\int_{\tau_{n_0}}^{\tau} h(\widetilde{\lambda}_n(\tau-\tau_1))\nabla\widetilde{v}_n d\tau_1[\nabla\Phi\widetilde{v}_n]dzd\tau \\
&+\frac{\widetilde{\lambda}_n}{2}\int_{2B}\left|\int_{\tau_{n_0}}^{1} h(\widetilde{\lambda}_n(1-\tau_1))\nabla\widetilde{v}_n d\tau_1\right|^2\psi(z)dz \\
&+\widetilde{\lambda}_n^2(\beta+1)\int_{-1}^{1}\left|\int_{\tau_{n_0}}^{\tau} h(\widetilde{\lambda}_n(\tau-\tau_1)\widetilde{\lambda}_n)\nabla\widetilde{v}_n d\tau_1\right|^2\Phi dzd\tau \\
&+\int_{-1}^{1}\int_{2B} h_{n_0}\nabla\widetilde{v}_{n_{00}}[\nabla\Phi\widetilde{v}_n+\Phi\nabla\widetilde{v}_n]dzd\tau-\int_{-1}^{1}\int_{2B}|\widetilde{v}_n|^{p+1}\Phi dzd\tau.
\end{aligned}
$$

Observe that $h(1)\leq h(\tau)$ and denote by $h'(\tau,\tau_1):=h((\tau-\tau_1)\widetilde{\lambda}_n)$, we have

$$
\begin{aligned}
&\widetilde{\lambda}_n\int_{2B}\left|\int_{\tau_{n_0}}^{1} h(\widetilde{\lambda}_n(1-\tau_1))\nabla\widetilde{v}_n d\tau_1\right|^2\psi(z)dz \\
\leq\ & \frac{3}{4}\widetilde{\lambda}_n\int_{-\frac{2}{3}}^{\frac{2}{3}}\int_{2B}\left|\int_{\tau_{n_0}}^{1} h(\widetilde{\lambda}_n(\tau-\tau_1))\nabla\widetilde{v}_n d\tau_1\right|^2\psi(z)dzd\tau \\
=\ & \frac{3}{4}\widetilde{\lambda}_n\int_{-\frac{2}{3}}^{\frac{2}{3}}\int_{2B}\left|\int_{\tau_{n_0}}^{\tau} h(\widetilde{\lambda}_n(\tau-\tau_1))\nabla\widetilde{v}_n d\tau_1\right|^2\psi(z)dzd\tau \\
&+\frac{3}{4}\widetilde{\lambda}_n\int_{-\frac{2}{3}}^{\frac{2}{3}}\int_{2B}\left|\int_{\tau}^{1} h(\widetilde{\lambda}_n(\tau-\tau_1))\nabla\widetilde{v} d\tau_1\right|^2\psi(z)dzd\tau \\
&+\frac{3}{2}\widetilde{\lambda}_n\int_{-\frac{2}{3}}^{\frac{2}{3}}\int_{2B}\left|\int_{\tau_{n_0}}^{\tau} h'(\tau,\tau_1)\nabla\widetilde{v}_n d\tau_1\int_{\tau}^{1} h'(\tau,\tau_1)\nabla\widetilde{v} d\tau_1\right|\psi(z)dzd\tau \\
\leq\ & \frac{3}{4}I_0+C\widetilde{\lambda}_n\int_{2B}\int_{-1}^{1}|\nabla\widetilde{v}|^2 d\tau_1 dz \\
&+C\widetilde{\lambda}_n I_0^{\frac{1}{2}}\left[\int_{2B}\int_{-1}^{1}|\nabla\widetilde{v}|^2 d\tau_1 dz\right]^{\frac{1}{2}} \\
=\ & \frac{3}{4}I_0+A,
\end{aligned}
$$

where

$$
I_0:=\widetilde{\lambda}_n\int_{-\frac{2}{3}}^{\frac{2}{3}}\int_{2B}\left|\int_{\tau_{n_0}}^{\tau} h(\widetilde{\lambda}_n(\tau-\tau_1))\nabla\widetilde{v}_n d\tau_1\right|^2\psi(z)dzd\tau
$$

and

$$\begin{aligned} A &:= C\widetilde{\lambda}_n \int_{2B}\int_{-1}^{1} |\nabla \widetilde{v}_n|^2 d\tau dz \\ & \quad + C I_0^{\frac{1}{2}} \left[\int_{2B}\int_{-1}^{1} |\nabla \widetilde{v}_n|^2 d\tau_1 dz\right]^{\frac{1}{2}} \to 0. \end{aligned}$$

Finally, using Cauchy-Schwartz inequality and Lemma 3.3, we obtain

$$\begin{aligned} \frac{1}{16} &\leq \frac{1}{2}(1-\frac{3}{4})I_0 \\ &\leq A + C\|\Phi\|_\infty (\int_{-1}^{1}\int_{2B} (\widetilde{v}_n)^2) + \|\Phi\|_\infty \int_{-1}^{1}\int_{2B} |\widetilde{v}_n|^{p+1} dz d\tau \\ & \quad + C\|\Phi\|_\infty \left[\left[\int_{-1}^{1}\int_{2B} [(\widetilde{v}_n)_\tau]^2\right]^{\frac{1}{2}} + \left[\int_{-1}^{1}\int_{2B} |\nabla \widetilde{v}_n|^2\right]^{\frac{1}{2}}\right] \left[\int_{-1}^{1}\int_{2B} \widetilde{v}_n^2\right]^{\frac{1}{2}} \\ & \quad + \widetilde{\lambda}_n \|\nabla\Phi\|_\infty \left[\int_{-1}^{1}\int_{2B} \widetilde{v}_n^2\right]^{\frac{1}{2}} \left[\int_{-1}^{1}\int_{2B} \left|\int_{\tau_{n_0}}^{\tau} h(\widetilde{\lambda}_n(\tau-\tau_1))\nabla \widetilde{v}_n d\tau_1\right|^2\right]^{\frac{1}{2}} \\ & \quad + \widetilde{\lambda}_n^2 (\beta+1)\|\Phi\|_\infty \int_{-1}^{1}\int_{2B} \left|\int_{\tau_{n_0}}^{\tau} h(\widetilde{\lambda}_n(\tau-\tau_1))\nabla \widetilde{v}_n d\tau_1\right|^2 \\ & \quad + h_{n_0}(-1) \left[\int_{-1}^{1}\int_{2B} |\nabla \widetilde{v}_{n_{00}}|^2\right]^{\frac{1}{2}} \\ & \qquad \times \left[\|\Phi\|_\infty \left[\int_{-1}^{1}\int_{2B} |\nabla \widetilde{v}_n|^2\right]^{\frac{1}{2}} + \|\nabla\Phi\|_\infty \left[\int_{-1}^{1}\int_{2B} \widetilde{v}_n^2\right]^{\frac{1}{2}}\right] \to 0. \end{aligned}$$

Which is a contradiction.
Case B: There exists a constant M such that

$$\lambda_n \sup_{a\in\mathbb{R}^n} \int_{\frac{1}{3}}^{\frac{2}{3}} \int_{2B} \left|\int_{\tau_{n_0}}^{\tau} h((\tau-\tau_1)\lambda_n)|\nabla v_n(\tau_1,z)|d\tau_1\right|^2 dz d\tau \leq M$$

As $v_n(\tau) = v_{n,a_0}(\tau)$ can be computed from any v_{n,a_0} by space translation: $v_n(\tau,z) = v_{n,a_0}(\tau, z + \frac{a_n - a_0}{\sqrt{\lambda_n}} e^{\frac{s_n+\lambda_n\tau}{2}})$, Lemma 3.2 implies that $v_n = v_{n,a_n}$ remains bounded in $H^1([\frac{1}{3},\frac{2}{3}]\times\mathbb{R}^N)$. As $H^1_{loc,u}([\frac{1}{3},\frac{2}{3}]\times\mathbb{R}^N)$ is compactly embedded into $L^p_{loc}([\frac{1}{3},\frac{2}{3}]\times\mathbb{R}^N)$ (because $p < 1+\frac{4}{N} \leq \frac{2(N+1)}{N-1}$, there exists a subsequence (also denoted v_n) such that $v_n \rightharpoonup V$ in $H^1_{loc,u}([\frac{1}{3},\frac{2}{3}]\times\mathbb{R}^N)$ and $v_n \to V$ in $L^p_{loc,u}([\frac{1}{3},\frac{2}{3}]\times\mathbb{R}^N)$ hence $|v_n|^{p-1}v_n \to |V|^{p-1}V$ in $L^1_{loc,u}([\frac{1}{3},\frac{2}{3}]\times\mathbb{R}^N)$, so in $D'([\frac{1}{3},\frac{2}{3}]\times\mathbb{R}^N)$.

Note that v_n is the solution of

$$
\begin{aligned}
&(v_n)_\tau + \lambda_n\beta(v_n) + \lambda_n \tfrac{z_n}{2}\nabla v_n - \lambda_n \textstyle\int_{\tau_{n_0}}^{\tau} h(\lambda_n(\tau-\tau_1))\Delta v_n d\tau_1 \\
&\qquad\qquad\qquad -h(s_n+\lambda_n\tau_n - s_0)\Delta v_{n_{00}} = |v_n|^{p-1}v_n.
\end{aligned}
\tag{27}
$$

Since

$$
\begin{aligned}
&\lambda_n \left| \int_{\frac{1}{3}}^{\frac{2}{3}} \int_{B(0,\frac{1}{\lambda_n})} \int_{\tau_{n_0}}^{\tau} h((\tau-\tau_1)\lambda_n)\Delta v_n(\tau_1,z)d\tau_1 v_n dz d\tau \right| \\
&= \lambda_n \left| \int_{\frac{1}{3}}^{\frac{2}{3}} \int_{B(0,\frac{1}{\lambda_n})} \int_{\tau_{n_0}}^{\tau} h((\tau-\tau_1)\lambda_n)\nabla v_n(\tau_1,z)d\tau_1 \nabla v_n dz d\tau \right| \\
&\le \sqrt{\lambda_n}\left[\lambda_n \int_{\frac{1}{3}}^{\frac{2}{3}} \int_{B(0,\frac{1}{\lambda_n})} \left| \int_{\tau_{n_0}}^{\tau} h((\tau-\tau_1)\lambda_n)\nabla v_n(\tau_1,z)d\tau_1 \right|^2\right]^{\frac{1}{2}} \\
&\qquad \times \left[\int_{\frac{1}{3}}^{\frac{2}{3}} \int_{B(0,\frac{1}{\lambda_n})} |\nabla v_n|^2 dz d\tau\right]^{\frac{1}{2}} \\
&\le \sqrt{\lambda_n}M\left[\int_{B(0,\frac{1}{\lambda_n})} |\nabla v_n|^2 dz d\tau\right]^{\frac{1}{2}} \le \sqrt{\lambda_n}M' \to 0,
\end{aligned}
$$

A passage to the limit (using Lemma 3.2) implies $|V|^{p-1}V = 0$ for all $(t,x) \in [\frac{1}{3},\frac{2}{3}] \times \mathbb{R}^N$. Therefore, $V = 0$. Let's prove that $v_n(0,z) \to 0$ in $L^2(B)$. Indeed,

$$
\begin{aligned}
\|v_n(0,z)\|^2_{L^2(B)} &= 3\int_{\frac{1}{3}}^{\frac{2}{3}} \int_B v_n(0,z)^2 dz\, d\tau \\
&\le 6\int_{\frac{1}{3}}^{\frac{2}{3}} \int_B [(v_n(0,z)-v_n(\tau,z))^2 + v_n(\tau,z)^2] dz d\tau \\
&\le 3\sup_{z\in B}\int_B [v_n(0,z)-v_n(\tau,z)]^2 dz \\
&\quad +6\left[\int_{\frac{1}{3}}^{\frac{2}{3}} \int_B (v_n(\tau,z))^{p+1} dz d\tau\right]^{\frac{2}{p+1}} (1/3)^{\frac{2p}{p+1}}.
\end{aligned}
$$

Lemma 3.2 and the relation (22) implies that $\frac{1}{2} \le \|v_n(0,z)\|_{L^2(B)} \to 0$. But this is a contradiction, since $\frac{1}{2} \le \|v_n(0,z)\|_{L^2(B)}$ (by (23)). This ends the proof of Proposition 3.2

Proof of Proposition 3.3. We proceed by contradiction. Suppose that there exists a sequence $\alpha_n \to \infty$ for all $a \in \mathbb{R}^N$, such that

$$
\int_{\alpha_n+\frac{1}{3}}^{\alpha_n+\frac{2}{3}} \int_B \left| \int_{s_0}^{s} h(s-s')\nabla w_a(s',y)ds' \right|^2 dy ds \to \infty.
$$

From proposition 3.2, we have

$$\sup_{\substack{a \in \mathbb{R}^N \\ s \geq s_0}} \int_B w_a^2(s,y)dy \leq M.$$

Define for any $a_0 \in \mathbb{R}^N$,

$$\widetilde{v}_{n,a_0}(\tau,z) := \widetilde{\lambda}_n^{\beta} w_{a_0,\tau_0}(s_n + \tau_0 + \widetilde{\lambda}_n \tau,\ \widetilde{\lambda}_n^{\frac{1}{2}} z).$$

More particularly, $\widetilde{v}_n := \widetilde{v}_{n,b_n,\tau_n}$, then choose $\lambda_n > 0$, $b_n \in \mathbb{R}^N$ in such a way that

$$\begin{aligned} \frac{1}{2} &\leq \widetilde{\lambda}_n \int_0^1 \int_B \left| \int_{\tau_{n_0}}^{\tau} h((\tau - \tau_1)\widetilde{\lambda}_n) \nabla \widetilde{v}_n(\tau_1, z) d\tau_1 \right|^2 dz d\tau \\ &\leq \sup_{\substack{a_0 \in \mathbb{R}^N \\ \tau_0 \in [\frac{1}{3}, \frac{2}{3}]}} \widetilde{\lambda}_n \int_0^1 \int_B \left| \int_{\tau_{n_0}}^{\tau} h((\tau - \tau_1)\widetilde{\lambda}_n) \nabla \widetilde{v}_{n,a_0,\tau_0}(\tau_1, z) d\tau_1 \right|^2 dz d\tau = 1, \end{aligned}$$

with $s_n + \tau_0 + \widetilde{\lambda}_n \tau_{n_0} = s_0$.
This choice is possible for all $0 < \widetilde{\lambda} \leq \frac{1}{3}$ because $h((\tau - \tau_1)\widetilde{\lambda}) = h'(\tau, \tau_1)$ and

$$\begin{aligned} &\sup_{\substack{a_0 \in \mathbb{R}^N \\ \tau_0 \in [\frac{1}{3},\ \frac{2}{3}]}} \widetilde{\lambda}^{2\beta+1} \int_0^1 \int_B \left| \int_{\tau_{n_0}}^{\tau} h'(\tau, \tau_1) \nabla w_{a_0,\tau_0}(s_n + \tau_0 + \widetilde{\lambda}\tau_1, \widetilde{\lambda}^{\frac{1}{2}} z) d\tau_1 \right|^2 dz d\tau \\ &\qquad \leq C \widetilde{\lambda}^{2\beta - \frac{N}{2}} \int_{s_n + \frac{1}{3}}^{s_n + \frac{5}{3}} \int_B \left| \int_{s_n + \tau_0 + \widetilde{\lambda}\tau_{n_0}}^{s} h((s - s_1)) \nabla w_{a_0}(s_1,\ y) ds_1 \right|^2 dy ds, \end{aligned}$$

where we have made, successively, the change of variables: $s_1 = s_n + \tau_0 + \widetilde{\lambda}\tau_1$ and then $s = s_n + \tau_0 + \tau$.

By the same calculation as in the proof of proposition 3.2, $\widetilde{v}_n$ is also a solution of (25) that satisfies

$$\frac{1}{2} \leq \widetilde{\lambda}_n \int_{-1}^1 \int_{2B} \left| \int_{\tau_{n_0}}^{\tau} h(\widetilde{\lambda}_n(\tau - \tau_1)) \nabla \widetilde{v}_n d\tau_1 \right|^2 \Phi_\tau dz d\tau \to 0,$$

which is a contradiction. □

Proof of Proposition 3.4. We proceed by contradiction. Suppose that there exists a sequence $s_n \to \infty$ such that, for all $a \in \mathbb{R}^n$,

$$\int_B \left(\int_{s_0}^{s_n} w_a(s,y) ds \right)^2 dy \to \infty.$$

By propositions 3.2 and 3.3, the family w_a satisfies

$$\sup_{\substack{a \in \mathbb{R}^N \\ s \geq s_0}} \int_B w_a^2(s,y)dy + \sup_{\substack{a \in \mathbb{R}^N \\ \alpha \geq s_0}} \int_\alpha^{\alpha+1} \int_B \left| \int_{s_0}^s h(s-s')\nabla w_a(s',y)ds' \right|^2 dyds \leq M.$$

We can then define the family of functions

$$v_{n,a_0}(\tau,z) := \lambda_n^\beta w_{n,a_0}(s_n + \lambda_n \tau, \sqrt{\lambda_n} z),$$

and more particularly $v_n = v_{n,a_n}$, where $\lambda_n > 0$ and $a_n \in \mathbb{R}^N$ are chosen in such a way that

$$\begin{aligned} \frac{1}{2} &\leq \lambda_n^2 \int_B \left[\int_{\tau_{n_0} = \frac{s_0 - s_n}{\lambda_n}}^0 v_n(\tau,z)d\tau \right]^2 dz \\ &\leq \lambda_n^2 \sup_{a_0 \in \mathbb{R}^N} \int_B \left[\int_{\tau_{n_0} = \frac{s_0 - s_n}{\lambda_n}}^0 v_{n,a_0}(\tau,z)d\tau \right]^2 dz = 1. \end{aligned} \tag{28}$$

This choice is possible for $0 < \lambda << 1$ because we have

$$\begin{aligned} \lambda^{2\beta+2} \sup_{a_0 \in \mathbb{R}^N} \int_B \left[\int_{\tau_{n_0} = \frac{s_0 - s_n}{\lambda}}^0 w_{n,a_0}(s_n + \lambda\tau, \sqrt{\lambda} z)d\tau \right]^2 dz \\ \leq \lambda^{2\beta - \frac{N}{2}} \sup_{a_0 \in \mathbb{R}^N} \int_B \left[\int_{s_0}^{s_n} w_{n,a_0}(s,y)ds \right]^2 dy. \end{aligned}$$

and $2\beta - \frac{N}{2} \geq 0$.
Note that v_n is the solution of

$$\begin{aligned} (v_n)_\tau + \lambda_n \beta v_n + \lambda_n \tfrac{z_n}{2} \nabla v_n - \lambda_n \textstyle\int_{\tau_{n_0}}^\tau h(\lambda_n(\tau - \tau_1))\Delta v_n d\tau_1 \\ -h_{n_0} \Delta v_{n00} = |v_n|^{p-1} v_n. \end{aligned} \tag{29}$$

Here $h_{n_0}(\tau) := h(s_n + \lambda_n \tau - s_0)$. Multiplying equation (29) by $\lambda_n \Phi \int_{\tau_{n_0}}^\tau v_n d\tau_1 := \lambda_n \int_{-1}^\tau \varphi_0(\tau_1)d\tau_1 \psi^p(z) \int_{\tau_{n_0}}^\tau v_n d\tau_1$ and then integrating over $[-1,1] \times 2B$, we obtain

$$\begin{aligned} \lambda_n \int_{-1}^1 \int_{2B} (v_n)_\tau \Phi \int_{\tau_{n_0}}^\tau v_n d\tau_1 dz d\tau + \frac{\lambda_n^2 \beta}{2} \int_{-1}^1 \int_{2B} \frac{d}{d\tau} \left[\Phi \left[\int_{\tau_{n_0}}^\tau v_n d\tau_1 \right]^2 \right] dz d\tau \\ -\frac{\lambda_n^2 \beta}{2} \int_{-1}^1 \int_{2B} \Phi_t \left[\int_{\tau_{n_0}}^\tau v_n d\tau_1 \right]^2 dz d\tau + \lambda_n^2 \int_{-1}^1 \int_{2B} \frac{z_n}{2} \nabla v_n \Phi \int_{\tau_{n_0}}^\tau v_n d\tau_1 dz d\tau \\ -\lambda_n^2 \int_{-1}^1 \int_{2B} \int_{\tau_{n_0}}^\tau h(\lambda_n(\tau - \tau_1))\Delta v_n d\tau_1 \Phi \int_{\tau_{n_0}}^\tau v_n d\tau_1 dz d\tau \end{aligned}$$

$$-\lambda_n \int_{-1}^{1}\int_{2B}(h_{n_0}\Delta v_{n_{00}}\Phi\int_{\tau_{n_0}}^{\tau} v_n d\tau_1)dzd\tau$$

$$= \lambda_n \int_{-1}^{1}\int_{2B}|v_n|^{p-1}v_n\Phi\int_{\tau_{n_0}}^{\tau} v_n d\tau_1 dzd\tau,$$

then

$$\lambda_n \int_{-1}^{1}\int_{2B}(v_n)_\tau\Phi\int_{\tau_{n_0}}^{\tau} v_n d\tau_1\ dzd\tau + \frac{\lambda_n^2\beta}{2}\int_{2B}\psi\left[\int_{\tau_{n_0}}^{1} v_n d\tau_1\right]^2 dz$$

$$-\frac{\lambda_n^2\beta}{2}\int_{-1}^{1}\int_{2B}\Phi_t\left[\int_{\tau_{n_0}}^{\tau} v_n d\tau_1\right]^2 dzd\tau + \lambda_n^2\int_{-1}^{1}\int_{2B}\frac{z_n}{2}\nabla v_n\Phi\int_{\tau_{n_0}}^{\tau} v_n d\tau_1 dzd\tau$$

$$+\lambda_n^2\int_{-1}^{1}\int_{\tau_{n_0}}^{\tau} h(\lambda_n(\tau-\tau_1))\nabla v_n d\tau_1\left[\nabla\Phi\int_{\tau_{n_0}}^{\tau} v_n d\tau_1 + \Phi\int_{\tau_{n_0}}^{\tau}\nabla v_n d\tau_1\right] dzd\tau$$

$$+\lambda_n\int_{-1}^{1}\int_{2B} h_{n_0}\nabla v_{n_{00}}\left[\nabla\Phi\int_{\tau_{n_0}}^{\tau} v_n d\tau_1 + \nabla\Phi\int_{\tau_{n_0}}^{\tau}\nabla v_n d\tau_1\right] dzd\tau$$

$$= \lambda_n \int_{-1}^{1}\int_{2B}|v_n|^{p-1}v_n\Phi\int_{\tau_{n_0}}^{\tau} v_n d\tau_1 dzd\tau.$$

Using Cauchy-Schwartz inequality and Lemma 3.2, we get

$$\begin{aligned}
\frac{\beta}{4} &\le \frac{\lambda_n^2\beta}{2}\int_{2B}\psi\left[\int_{\tau_{n_0}}^{1} v_n d\tau_1\right]^2 dz \\
&\le \|\Phi\|_\infty\left[\lambda_n^2\int_{-1}^{1}\int_{2B}\left(\int_{\tau_{n_0}}^{\tau} v_n d\tau_1\right)^2 dzd\tau\right]^{\frac{1}{2}} \\
&\qquad \times\left[\left(\int_{-1}^{1}\int_{2B}(v_n)_\tau^2\, dzd\tau\right)^{\frac{1}{2}} + \lambda_n\left(\int_{-1}^{1}\int_{2B}|\nabla v_n d\tau_1|^2 dzd\tau\right)^{\frac{1}{2}}\right] \\
&+\left\{\lambda_n^2\left[\int_{-1}^{1}\int_{2B}\left(\int_{\tau_{n_0}}^{\tau} h(\tau-\tau_1)\nabla v_n d\tau_1\right)^2\right]^{\frac{1}{2}} + \lambda_n h_{n_0}(-1)\left(\int_{-1}^{1}\int_{2B}|\nabla v_{n_{00}}|^2\right)^{\frac{1}{2}}\right\} \\
&\qquad \times\left\{\|\nabla\Phi\|_\infty\left[\int_{-1}^{1}\int_{2B}\left(\int_{\tau_{n_0}}^{\tau} v_n d\tau_1\right)^2 dzd\tau\right]^{\frac{1}{2}}\right. \\
&\qquad\qquad \left. +\|\Phi\|_\infty\left[\int_{-1}^{1}\int_{2B}\left(\int_{\tau_{n_0}}^{\tau}\nabla v_n d\tau_1\right)^2 dzd\tau\right]^{\frac{1}{2}}\right\} \\
&\quad +\lambda_n^2\beta/2\int_{-1}^{1}\int_{2B}\Phi_t\left(\int_{\tau_{n_0}}^{\tau} v_n d\tau_1\right)^2 dzd\tau
\end{aligned}$$

$$+\left|\lambda_n \int_{-1}^{1}\int_{2B} |\psi v_n|^p \Phi \int_{\tau_{n_0}}^{\tau} v_n d\tau_1 dz d\tau\right| = I_1 + I_2 + I_3,$$

where

$$\begin{aligned}
I_2 &= \frac{\lambda_n^2\beta}{2}\int_{-1}^{1}\int_{2B}\Phi_t\left(\int_{\tau_{n_0}}^{\tau} v_n d\tau_1\right)^2 dz d\tau \\
&\le \frac{\lambda_n^2\beta}{2}C\int_{-1}^{1}\int_{2B}\left|\int_{\tau_{n_0}}^{\tau}\nabla(\Phi_t v_n)d\tau_1\right|^2 dz d\tau \\
&\le \frac{\lambda_n^2\beta C}{2}\int_{-1}^{1}\int_{2B}\left|\int_{\tau_{n_0}}^{\tau} h(\tau-\tau_1)\nabla(\Phi_\tau v_n)d\tau_1\right|^2 dz d\tau \to 0,
\end{aligned}$$

using Poincare's Inequality and proposition 3.3.

$$\begin{aligned}
I_3 &= \lambda_n \int_{-1}^{1}\int_{2B}|\psi v_n|^p \int_{\tau_{n_0}}^{\tau} v_n d\tau_1 dz d\tau \\
&\le \lambda_n\left[\int_{-1}^{1}\int_{2B}|\psi v_n|^{2p} d\tau dz\right]^{\frac{1}{2}}\left[\int_{-1}^{1}\int_{2B}\left|\int_{\tau_{n_0}}^{\tau} v_n d\tau_1\right|^2 d\tau dz\right]^{\frac{1}{2}} \\
&\le C\lambda_n\left[\int_{-1}^{1}\int_{2B}|\nabla[\psi v_n]|^2 d\tau dz\right]^{p}\left[\int_{-1}^{1}\int_{2B}\left|\int_{\tau_{n_0}}^{\tau} v_n d\tau_1\right|^2 d\tau dz\right]^{\frac{1}{2}} \to 0,
\end{aligned}$$

and

$$\begin{aligned}
I_1 \quad &= \quad \|\Phi\|_\infty\left[\lambda_n^2\int_{-1}^{1}\int_{2B}\left(\int_{\tau_{n_0}}^{\tau} v_n d\tau_1\right)^2 dz d\tau\right]^{\frac{1}{2}} \\
&\qquad \times\left[\left(\int_{-1}^{1}\int_{2B}(v_n)_\tau^2\, dz d\tau\right)^{\frac{1}{2}} + \lambda_n\left(\int_{-1}^{1}\int_{2B}|\nabla v_n d\tau_1|^2 dz d\tau\right)^{\frac{1}{2}}\right] \\
&+\left\{\lambda_n^2\left[\int_{-1}^{1}\int_{2B}\left(\int_{\tau_{n_0}}^{\tau} h(\tau-\tau_1)\nabla v_n d\tau_1\right)^2\right]^{\frac{1}{2}} + \lambda_n h_{n_0}(-1)\left(\int_{-1}^{1}\int_{2B}|\nabla v_{n_{00}}|^2\right)^{\frac{1}{2}}\right\} \\
&\qquad \times\left\{\|\nabla\Phi\|_\infty\left[\int_{-1}^{1}\int_{2B}\left(\int_{\tau_{n_0}}^{\tau} v_n d\tau_1\right)^2 dz d\tau\right]^{\frac{1}{2}}\right. \\
&\qquad\qquad \left. +\|\Phi\|_\infty\left[\int_{-1}^{1}\int_{2B}\left(\int_{\tau_{n_0}}^{\tau}\nabla v_n d\tau_1\right)^2 dz d\tau\right]^{\frac{1}{2}}\right\} \\
&\longrightarrow \quad 0 \qquad \text{(Using Lemma 3.2, Propositions 3.2 and 3.3)}
\end{aligned}$$

which is a contradiction. This ends the proof of Proposition 3.4.

4 Appendix A: Lower bound of $\|u_t\|_{L^\infty}$ for $N = 1$

In this section we will prove that $\kappa(T-t)^{-\beta}$ is an optimal bound for $N = 1$ in the L^∞ norm for a blowing-up solution. For this, we recall Duhamel's formula in the one-dimensional case (see [1])

$$2u(t,x) = u_0(x+t)+u_0(x-t)+\int_{x-t}^{x+t} u_1(\zeta)d\zeta+\int_0^t \int_{x-t+s}^{x+t-s} h(s,\zeta)d\zeta ds \quad (30)$$

for the wave equation

$$\begin{cases} u_{tt} - \Delta u = h \\ \\ u(x,0) = u_0, \;\; u_t(x,0) = u_1. \end{cases}$$

Proposition 4.1 *Assume $N = 1$ and u_t blows-up at time $T < \infty$, then*

$$\limsup_{t\to T^-}(T-t)^\beta \|u_t(t)\|_{L^\infty} \geq \kappa = \beta^\beta.$$

Proof. The proof will be done in two steps:
First step: Define

$$F(t) := \int_0^t \|u_s(s)\|^p_{L^\infty} ds.$$

Then

$$\liminf_{t\to T^-}(T-t)^\beta F(t) \geq \kappa. \quad (31)$$

Indeed, from (30), we have

$$\begin{aligned} u_t(t,x) \;\; &= \;\; \frac{1}{2}\frac{\partial}{\partial t}(u_0(x+t)+u_0(x-t)) + \frac{1}{2}(u_1(x+t) - u_1(x-t)) \\ &\quad + \frac{1}{2}\int_0^t \left[|u_s|^{p-1}u_s(s,x+t-s) - |u_s|^{p-1}u_s(s,x-t+s)\right] ds. \end{aligned}$$

Hence, $\forall t \in [0,T)$, we have

$$(F'(t))^{\frac{1}{p}} = \|u_t(t)\|_{L^\infty} \leq \|(u_0)_x\|_{L^\infty} + \|u_1\|_{L^\infty} + \int_0^t \|u_s(s)\|^p_{L^\infty} ds. \quad (32)$$

Then

$$(F'(t))^{\frac{1}{p}} \leq K + F(t) = F(t)[1+\varepsilon(t)], \quad (33)$$

with $\varepsilon(t) := K/F(t)$. This, from one side, implies that, for all $\tau \in [t,s] \subset [0,T)$

$$F'(\tau)F^{-p}(\tau) \leq= (1+\varepsilon(t))^p, \quad (34)$$

since ε is decreasing. Integrating (34) over $[t,s]$ with $s \in (t,T)$, making $s \uparrow T$ and using the fact that $\lim_{s\uparrow T} F(s) = \infty$, we get

$$F(t) \geq \kappa[(1+\varepsilon(t))(T-t)]^{\beta}$$

where $\varepsilon(t) \to 0$ as $t \uparrow T$, which concludes the proof of (31).

Second step: In order to prove the proposition, we argue by contradiction, assuming that

$$\|u_t(t)\|_{L^\infty} < \frac{\kappa(1-\delta)}{(T-t)^{\beta}} + C,$$

with $0<\delta<1$ and since $\beta p = \beta + 1$,

$$F(t) = \int_0^t \|u_\tau(\tau)\|^p_{L^\infty} d\tau < \frac{(\kappa(1-\delta))^p}{\beta(T-t)^{\beta}} + C'.$$

Then

$$(T-t)^{\beta} F(t) < \beta^{\beta}(1-\delta)^p + (T-t)^{\beta} C',$$

which contradicts (31). $\square$

5 Appendix B: Proof of Lemmas 3.2 and 3.3

Proof of Lemma 3.2.

i) We remark that, for $\tau, \tau' \in [0,1]$, and $a_0 \in \mathbb{R}^N$, we have, for $0 < \lambda_n \leq \frac{1}{16}$

$$\begin{aligned}
\|v_{n,a_0}(\tau) - v_{n,a_0}(\tau')\|^2_{L^2(2B)} &= \int_{2B} \left| \int_{\tau'}^{\tau} (v_{n,a_0}(\sigma,z))_\sigma \right|^2 d\sigma dz \\
&\leq (\tau-\tau') \int_{2B} \int_{\tau'}^{\tau} [(v_{n,a_0}(\sigma z))_\sigma]^2 d\sigma dz \\
&\leq \int_{2B} \int_0^1 [(v_{n,a_0}(\sigma,z))_\sigma]^2 d\sigma dz \\
&= \int_0^1 \int_{2B} [\lambda_n^{\beta+1} \frac{\partial w_{a_0}}{\partial s}(s_n + \lambda_n \sigma, \sqrt{\lambda_n} z)]^2 dz d\sigma \\
&\leq \lambda_n^{2(\beta+1)-(\frac{N}{2}+1)} \sup_{a_0\in\mathbb{R}^N} \int_{s_n}^{s_n+\lambda_n} \int_{2\sqrt{\lambda_n}B} \left[\frac{\partial w_a}{\partial s}(s,y)\right]^2 dyds \\
&\leq M_0 \lambda_n^{2\beta - \frac{N}{2}+1}.
\end{aligned}$$

ii)

$$\begin{aligned}
\int_0^1 \|\nabla v_{n,a_0}\|^2_{L^2(2B)} d\tau &= \int_0^1 \int_{2B} |\nabla(v_{n,a_0}(\tau,z))|^2 d\tau dz \\
&= \int_0^1 \int_{2B} \left| \lambda_n^{\beta} \nabla[w_{a_0}(s_n + \lambda_n \tau, \sqrt{\lambda_n} z)] \right|^2 dz d\tau
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^1 \int_{2B} \left| \lambda_n^{\beta+\frac{1}{2}} \nabla w_{a_0}(s_n+\lambda_n\tau, \sqrt{\lambda_n}z) \right|^2 dz d\tau \\
&\leq \lambda_n^{(2\beta+1)-(\frac{N}{2}+1)} \sup_{a_0\in\mathbb{R}^N} \int_{s_n}^{s_n+\lambda_n} \int_{2\sqrt{\lambda_n}B} |\nabla w_{a_0}(s,y)|^2 dy ds \\
&\leq M_0 \lambda_n^{2\beta-\frac{N}{2}}.
\end{aligned}
$$

By a similar calculations, one can find the other inequalities. □

Proof of Lemma 3.3: (a) This will be done in several steps.

1. First, notice that $\widetilde{\lambda}_n \to 0$ since $0 < \widetilde{\lambda}_n < \lambda_n \to 0$.

Now, we prove that $\frac{\widetilde{\lambda}_n}{\lambda_n} \to 0$. Assume that $\frac{\widetilde{\lambda}_n}{\lambda_n} \not\to 0$, so there exists $\eta_0 > 0$ such that $\forall n \in \mathbb{N}$, $\frac{\widetilde{\lambda}_n}{\lambda_n} > \eta_0$. We deduce from the construction of $\widetilde{\lambda}_n$, $\widetilde{\lambda}_n < \lambda_n < \frac{1}{3}$, that $\eta_0 < \frac{1}{3}$.

Now, Let $z_1, .., z_k \in 2B$, $\sigma_1, ..., \sigma_l \in [\frac{1}{3}, \frac{2}{3}]$, $2B \subset \cup_{i=1}^k B(z_i, \frac{\sqrt{\eta_0}}{2})$ and $[\frac{1}{3}, \frac{2}{3}] \subset \cup_{j=1}^l [\sigma_j - \frac{\eta_0}{3}, \sigma_j + \frac{\eta_0}{3}]$. Then

$$
\int_{\frac{1}{3}}^{\frac{2}{3}} \int_{2B} \left| \int_{\tau_{n_0}}^{\tau} h(\widetilde{\lambda}_n(\tau-\tau_1)) \nabla v_n d\tau_1 \right|^2 dz d\tau \leq \sum_{i,j} I_{i,j},
$$

where

$$
I_{i,j} = \int_{\sigma_j-\frac{\eta_0}{3}}^{\sigma_j+\frac{\eta_0}{3}} \int_{B(z_i,\frac{\sqrt{\eta_0}}{2})} \left| \int_{\tau_{n_0}}^{\tau'} h(\widetilde{\lambda}_n(\tau'-\tau_1)) \nabla v_n(\tau_1, z') d\tau_1 \right|^2 dz' d\tau'.
$$

Using the following change of variables: $\tau_2 = (\tau_1 - \sigma_j)\lambda_n/\widetilde{\lambda}_n$ and $z = (z'-z_i)\sqrt{\lambda_n/\widetilde{\lambda}_n} \in B(0, \frac{1}{2}\sqrt{\frac{\eta_0\lambda_n}{\widetilde{\lambda}_n}}) \subset \frac{B}{2}$, and since $\varphi_0 \geq 1$ in $[-\frac{2}{3}, \frac{2}{3}]$ and $\varphi_0 \geq 0$ in $[-1,1]$, we obtain

$$
\begin{aligned}
I_{i,j} \leq\ & (\widetilde{\lambda}_n/\lambda_n)^{\frac{N}{2}+2} \int_{\sigma_j-\frac{\eta_0}{3}}^{\sigma_j+\frac{\eta_0}{3}} \int_{\frac{B}{2}} \\
& \left| \int_{\frac{\lambda_n}{\widetilde{\lambda}_n}(\tau_{n_0}-\sigma_j)}^{\frac{\lambda_n}{\widetilde{\lambda}_n}(\tau'-\sigma_j)} h(\widetilde{\lambda}_n f(\tau')) \nabla v_{n,a_0}(\frac{\widetilde{\lambda}_n}{\lambda_n}\tau_2 + \sigma_j, z\sqrt{\frac{\widetilde{\lambda}_n}{\lambda_n}} + z_i) d\tau_2 \right|^2 dz d\tau',
\end{aligned}
$$

with $f(\tau') = \tau' - \frac{\widetilde{\lambda}_n}{\lambda_n}\tau_2 - \sigma_j$, Using the second change of variables

$$
\tau_3 = \lambda_n/\widetilde{\lambda}_n(\tau'-\sigma_j) \in \lambda_n/\widetilde{\lambda}_n[-\eta_0/3, \eta_0/3] \subset [-1/3, 1/3],
$$

we get

$$
\begin{aligned}
I_{i,j} \leq\ & (\widetilde{\lambda}_n/\lambda_n)^{\frac{N}{2}+2-2\beta} \int_{-1}^{1} \int_{\frac{B}{2}} \\
& \left| \int_{\frac{\lambda_n}{\widetilde{\lambda}_n}(\tau_{n_0}-\sigma_j)}^{\tau_3} h(\widetilde{\lambda}_n^2(\tau_3-\tau_2)/\lambda_n) \nabla \widetilde{v}_{n,a_0+\sqrt{z_i}\exp((-s_n-\lambda_n\sigma_j)2)}(\tau_1, z+\varepsilon_i) d\tau_2 \right|^2 dz d\tau_3,
\end{aligned}
$$

where $\varepsilon_i = -z_i(e^{\frac{\widetilde{\lambda}_n\tau}{2}}-1)\sqrt{\lambda_n/\widetilde{\lambda}_n}$. Since $e^{\widetilde{\lambda}_n\tau/2}-1 \to 0$, we have $\sup_{[-1,1]} \varepsilon_i = 0$, for all $\tau \in [-1,1]$. Thus $B(\varepsilon_i, \frac{1}{2}) \subset B$, for all i and this implies $I_{i,j}$ is bounded from the relation (24) which contradicts the hypothesis of the case A.
2.

$$\begin{aligned}\int_{-1}^{1}\int_{2B} |\widetilde{v}_{n\tau}|^2 dz d\tau &= \widetilde{\lambda}_n^{2(\beta+1)-(\frac{N}{2}+1)} \int_{s_n+\lambda_n\tau_0-\widetilde{\lambda}_n}^{s_n+\lambda_n\tau_0+\widetilde{\lambda}_n} \int_{2\sqrt{\widetilde{\lambda}_n}B} \left|\frac{\partial w_{bn}(s,y)}{\partial s}\right|^2 dyds \\ &\leq M\widetilde{\lambda}_n^{2\beta-(\frac{N}{2}-1)} \to 0.\end{aligned}$$

3. From (23) and Lemma (3.2)(i), we deduce that

$$\sup_{\substack{a\in\mathbb{R}^n \\ \tau\in[-1,1]}} \int_{2B} |v_{n,a_0}(\tau,z)|^2 dz \leq C,$$

thus for all $\tau \in [-1,1]$,

$$\begin{aligned}\int_{2B} |\widetilde{v}_n(\tau,z)|^2 dz &= (\widetilde{\lambda}_n/\lambda_n)^{2\beta-\frac{N}{2}} \int_{2\sqrt{\widetilde{\lambda}_n}B} |v_{n,b_n}(\tau_n + \frac{\widetilde{\lambda}_n}{\lambda_n}\tau, z)|^2 dy \\ &\leq M(\widetilde{\lambda}_n/\lambda_n)^{2\beta-\frac{N}{2}} \to 0,\end{aligned}$$

from (28) and Lemma 3.2(iii).
4. For all $\widetilde{\lambda}_n \leq \frac{1}{16}$, we have

$$\begin{aligned}\int_{-1}^{1}\int_{2B} |\nabla\widetilde{v}_n|^2 dz d\tau &= \widetilde{\lambda}_n^{2\beta+1-(\frac{N}{2}+1)} \int_{s_n+\lambda_n\tau_0-\widetilde{\lambda}_n}^{s_n+\lambda_n\tau_0+\widetilde{\lambda}_n} \int_{2\sqrt{\widetilde{\lambda}_n}B} |\nabla(w_{bn}(s,y))|^2 dyds \\ &\leq M\widetilde{\lambda}_n^{2\beta-\frac{N}{2}} \to 0.\end{aligned}$$

5. Similarly we find

$$\int_{-1}^{1}\int_{2B} |\widetilde{v}_n(\tau,z)|^{p+1}\, dz \leq M(\widetilde{\lambda}_n/\lambda_n)^{2\beta-N/2} \to 0.$$

(b) Let us fix p points $z_1, \cdots, z_p \in 2B \subset \cup_{i=1}^{p} B(z_i, \frac{1}{2})$. For n large enough, and since we have $B(z_i, \frac{1}{2}) \subset B(e^{\frac{\widetilde{\lambda}_n\tau}{2}} z_i, 1)$ for all $\tau \in [-1,1]$, we have, (using (24)),

$$\begin{aligned}&\widetilde{\lambda}_n \int_{-1}^{1}\int_{B(z_i,\frac{1}{2})} \varphi_0(\tau) \left|\int_{\tau_{n_0}}^{\tau} h((\tau-\tau_1)\widetilde{\lambda}_n)\nabla\widetilde{v}_n(\tau_1,z)d\tau_1\right|^2 dzd\tau \\ \leq\ & \lambda_n \int_{-1}^{1} \varphi_0(\tau) \int_{B(e^{\frac{\widetilde{\lambda}_n\tau}{2}} z_i,1)} \left|\int_{\tau_{n_0}}^{\tau} h((\tau-\tau_1)\widetilde{\lambda}_n)\nabla\widetilde{v}_n(\tau_1,z)d\tau_1\right|^2 dzd\tau\end{aligned}$$

$$
\begin{aligned}
&\leq \lambda_n \int_{-1}^{1} \varphi_0(\tau) \int_B \left| \int_{\tau_{n_0}}^{\tau} h((\tau-\tau_1)\widetilde{\lambda}_n) \nabla \widetilde{v}_n(\tau_1, z + e^{\frac{\widetilde{\lambda}_n \tau}{2}} z_i) d\tau_1 \right|^2 dz d\tau \\
&\leq \widetilde{\lambda}_n \int_{-1}^{1} \varphi_0(\tau) \int_B \left| \int_{\tau_{n_0}}^{\tau} h((\tau-\tau_1)\widetilde{\lambda}_n) \nabla \widetilde{v}_{n, b_n + \sqrt{\widetilde{\lambda}_n} e^{-\frac{s_n + \lambda_n \tau_n}{2}} z_i, \tau_n} (\tau_1, z) d\tau_1 \right|^2 dz d\tau \leq 1.
\end{aligned}
$$

From this, we deduce that

$$
\begin{aligned}
&\widetilde{\lambda}_n \int_{-1}^{1} \int_{2B} \varphi_0(\tau) \left| \int_{\tau_{n_0}}^{\tau} h((\tau-\tau_1)\widetilde{\lambda}_n) \nabla \widetilde{v}_n(\tau_1, z) d\tau_1 \right|^2 dz d\tau \\
\leq\ &\widetilde{\lambda}_n \sum_{i=1}^{p} \int_{-1}^{1} \int_{B(z_i, \frac{1}{2})} \varphi_0(\tau) \left| \int_{\tau_{n_0}}^{\tau} h((\tau-\tau_1)\widetilde{\lambda}_n) \nabla \widetilde{v}_n(\tau_1, z) d\tau_1 \right|^2 dz d\tau \leq p.
\end{aligned}
$$

(c) Using the same notations as in (b), we have

$$
\begin{aligned}
\lambda_n^2 \int_{2B} \left[\int_{\tau_{n_0} = \frac{s_0 - s_n}{\lambda_n}}^{\tau} v_n(\tau, z) d\tau \right]^2 dz &\leq \lambda_n^2 \sum_{1 \leq i \leq p} \int_{B(z_i, \frac{1}{2})} \left[\int_{\tau_{n_0} = \frac{s_0 - s_n}{\lambda_n}}^{\tau} v_n(\tau, z) d\tau \right]^2 dz \\
&\leq \lambda_n^2 \sum_{1 \leq i \leq p} \int_B \left[\int_{\tau_{n_0} = \frac{s_0 - s_n}{\lambda_n}}^{1} v_n(\tau, z + z_i) d\tau \right]^2 dz \\
&\leq \lambda_n^2 \sum_{1 \leq p \leq n} \int_B \left[\int_{\tau_{n_0} = \frac{s_0 - s_n}{\lambda_n}}^{0} v_n(\tau, z) d\tau + \int_0^1 v_n(\tau, z) d\tau \right]^2 dz \leq C,
\end{aligned}
$$

using the hypotesis (28) and Lemma 3.2(iii). □

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